



CENTRAL EXTENSIONS OF NON-ASSOCIATIVE K -ALGEBRAS

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Résumé. Soit K un anneau commutatif unitaire. Par K -algèbre, nous entendons tout K -module A muni d'un homomorphisme de K -modules $A \otimes_K A \rightarrow A$, noté $a \otimes b \mapsto ab$. Nous considérons la variété Alg_K de toutes les K -algèbres, ainsi que sa sous-variété Alg_K^{AC} des K -algèbres à la fois associatives et commutatives. Nous montrons qu'un épimorphisme régulier $f : A \rightarrow B$ dans Alg_K est une extension centrale relativement à Alg_K^{AC} si et seulement si le noyau de f est contenu dans le centre de A , convenablement défini.

Abstract. Let K be a commutative ring with identity. By a K -algebra we mean any K -module A equipped with a K -module homomorphism $A \otimes_K A \rightarrow A$ written as $a \otimes b \mapsto ab$. We consider the variety Alg_K of all K -algebras, and its subvariety Alg_K^{AC} of K -algebras that are both associative and commutative. We show that a regular epimorphism $f : A \rightarrow B$ in Alg_K is a central extension with respect to Alg_K^{AC} if and only if the kernel of f is contained in the suitably defined centre of A .

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1. Introduction

For a fixed commutative ring K with identity, we consider the reflection $I : \text{Alg}_K \rightarrow \text{Alg}_K^{\text{AC}}$ from the variety Alg_K of K -algebras (by a K -algebra we mean any K -module A equipped with a K -module homomorphism $A \otimes_K$

$A \rightarrow A$ written as $a \otimes b \mapsto ab$) to the subvariety Alg_K^{AC} of associative and commutative K -algebras. For an algebra A in Alg_K , we denote by $R(A)$ the kernel of the canonical epimorphism $\eta_A : A \rightarrow I(A)$, and by $Z(A)$ the *centre* of A , which is defined as follows:

Definition 1.1. The *centre* $Z(A)$ of a K -algebra A is the intersection of the sets

$$\{a \in A \mid \forall_{x \in A} [a, x] = 0\}$$

and

$$\{a \in A \mid \forall_{x \in A}, \forall_{y \in A} \langle a, x, y \rangle = \langle x, a, y \rangle = \langle x, y, a \rangle = 0\},$$

where $[x, y] = xy - yx$ is the *commutator*, and $\langle x, y, z \rangle = (xy)z - x(yz)$ the *associator*.

This definition of centre is the one used by non-associative algebraists, as shown e.g. in [5]. It is useful in the sense that it generalizes both the “commutative” centre used with associative algebras, as well as the “associative” centre used for example with Jordan algebras.

Using this, the following definition is then a non-associative generalization of Definition 1.1 in [3]:

Definition 1.2. A regular epimorphism (equivalently, a surjective homomorphism) $f : A \rightarrow B$ of K -algebras is said to be:

1. *classically-central* if $\text{Ker}(f) \subseteq Z(A)$;
2. *algebraically-central* if any two parallel morphisms u and v , both with codomain A , satisfy

$$fu = fv \Rightarrow R(u) = R(v);$$

3. *categorically-central* if the diagram

$$\begin{array}{ccc} A \times_B A & \xrightarrow{\eta_{A \times_B A}} & I(A \times_B A) \\ \pi \downarrow & & \downarrow I(\pi) \\ A & \xrightarrow{\eta_A} & I(A) \end{array}$$

in which π is either of the two pullback projections $A \times_B A \rightarrow A$, is a pullback.

It was already shown in [4] (in particular, see Theorem 5.2) that, in the more general context of varieties of Ω -groups, f is algebraically-central if and only if it is categorically-central. The purpose of this paper is then to show that f is classically-central if and only if it is algebraically-central (this is Theorem 2.5), thus tying this more general definition of classical centrality to Janelidze and Kelly's well-developed categorical theory (and its algebraic predecessor, mentioned in [4]; see also [2]). We do this "directly" (i.e. using algebraic identities), but one might also try several "less direct" approaches, such as using split extensions (as done by Janelidze in [3]) or using relative commutator theory (as introduced by Everaert in [2]).

Remark 1.3. Since Alg_K is a variety of Ω -groups, it easily follows from Theorem 5.2 of [4] that a surjective homomorphism $f : A \rightarrow B$ of K -algebras is algebraically-central if and only if

$$(x, y) \in R(A \times_B A) \Rightarrow x = y \quad (\dagger)$$

for all $x, y \in A$.

In fact, the same is true in the general context of [4], and even more generally for arbitrary semi-abelian varieties of universal algebras (where one can use Theorem 2.1 of [1] instead of Theorem 5.2 of [4]).

2. Central Extensions of K -Algebras

For our proof of Theorem 2.5, it is also necessary that we define the reflection I explicitly: for an algebra A , $R(A)$ is given by the ideal generated by all elements of the forms $[x, y]$ and $\langle x, y, z \rangle$ where $x, y, z \in A$. This makes $I(A) = A/R(A)$ associative and commutative, since $I(A) = Z(I(A))$.

The main result we will need is the fact that if $\text{Ker}(f) \subseteq Z(A)$, then $R(A \times_B A)$ is a subalgebra of the diagonal 1_A (where $A \times_B A$ is the kernel-congruence of f); this is shown in Lemmas 2.1 to 2.4. So, for the duration of these proofs, let $\text{Ker}(f) \subseteq Z(A)$, and note the following:

Lemma 2.1. For any $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A \times_B A$,

1. $[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], [x_1, x_2])$, and
2. $\langle (x_1, y_1), (x_2, y_2), (x_3, y_3) \rangle = (\langle x_1, x_2, x_3 \rangle, \langle x_1, x_2, x_3 \rangle)$.

Proof. We first rewrite each of the elements as $(x_1, x_1 + k_1)$, $(x_2, x_2 + k_2)$, and $(x_3, x_3 + k_3)$ for some $k_1, k_2, k_3 \in \text{Ker}(f)$. Applying the commutator yields

$$[(x_1, x_1 + k_1), (x_2, x_2 + k_2)] = ([x_1, x_2], [x_1 + k_1, x_2 + k_2]).$$

Then, a rudimentary calculation gives

$$\begin{aligned} & [x_1 + k_1, x_2 + k_2] \\ &= [x_1, x_2] + [x_1, k_2] + [k_1, x_2] + [k_1, k_2] \quad (\text{the commutator is multilinear}) \\ &= [x_1, x_2] \quad (\text{since } \text{Ker}(f) \subseteq Z(A)). \end{aligned}$$

On the other hand, applying the associator, we get

$$\begin{aligned} & \langle (x_1, x_1 + k_1), (x_2, x_2 + k_2), (x_3, x_3 + k_3) \rangle \\ &= (\langle x_1, x_2, x_3 \rangle, \langle x_1 + k_1, x_2 + k_2, x_3 + k_3 \rangle). \end{aligned}$$

Then, a similar simple calculation using the multilinearity of the associator (and the fact that $\text{Ker}(f) \subseteq Z(A)$) yields

$$\langle x_1 + k_1, x_2 + k_2, x_3 + k_3 \rangle = \langle x_1, x_2, x_3 \rangle$$

which completes the proof. □

Next, consider the set

$$N(A, f) = \{x \in A \mid k \in \text{Ker}(f) \Rightarrow kx = 0 = xk\}$$

and note the following:

Lemma 2.2. $N(A, f)$ is an ideal in A .

Proof. For $x \in N(A, f)$ and $a \in A$ we have

$$\begin{aligned} k(ax) &= (ka)x = (ak)x = a(kx) = a0 = 0, \\ (ax)k &= a(xk) = a0 = 0; \end{aligned}$$

similarly, $k(xa) = 0 = (xa)k$. Since $N(A, f)$ is (clearly) closed under addition and scalar multiplication, this completes the proof. □

Lemma 2.3. $R(A) \subseteq N(A, f)$.

Proof. Since $N(A, f)$ is an ideal in A (Lemma 2.2), it suffices to show that $[x, y]$ and $\langle x, y, z \rangle$ belong to $N(A, f)$ for every $x, y, z \in A$. For $k \in \text{Ker}(f)$, we have

$$\begin{aligned} k[x, y] &= k(xy - yx) = (kx)y - y(kx) = 0, \text{ and} \\ k\langle x, y, z \rangle &= k((xy)z - x(yz)) = ((kx)y)z - (kx)(yz) = 0 \end{aligned}$$

since $kx \in \text{Ker}(f) \subseteq Z(A)$. Similarly, $[x, y]k = 0$ and $\langle x, y, z \rangle k = 0$. $\square$

Lemma 2.4. $(\dagger)$ holds; that is, $R(A \times_B A) \subseteq 1_A$, where 1_A is the diagonal (see Remark 1.3).

Proof. Let $S = R(A \times_B A) \cap 1_A$. By Lemma 2.1, S contains the canonical generators of $R(A \times_B A)$, and so it would suffice to prove that S is an ideal in $A \times_B A$. Since $R(A \times_B A)$ is an ideal in $A \times_B A$, we only have to show that

$$\begin{aligned} &((x, x) \in R(A \times_B A) \ \& \ k \in \text{Ker}(f)) \\ \Rightarrow &(y, y + k)(x, x), (x, x)(y, y + k) \in 1_A \quad (*) \end{aligned}$$

for every $y \in A$. Since η is a natural transformation, (x, x) belongs to $R(A \times_B A)$ if and only if x belongs to $R(A)$. Therefore, to prove $(*)$ is equivalent to proving the implication

$$(x \in R(A) \ \& \ k \in \text{Ker}(f)) \Rightarrow kx = 0 = xk,$$

which follows immediately from Lemma 2.3. $\square$

We are now in a position to prove the main result, using the results above and some results from [4]. In fact, the structure of the proof is also lifted from [4]: see there in particular the example after Theorem 5.2. We simply apply it to this new context, although with more results required (hence the above lemmas).

Theorem 2.5. Let $f : A \rightarrow B$ be a regular epimorphism. f is classically-central if and only if it is algebraically-central.

Proof. ($\Rightarrow$) Say f is classically-central, i.e. $\text{Ker}(f) \subseteq Z(A)$. Lemma 2.4 gives that $(\dagger)$ is satisfied.

($\Leftarrow$) Say f is algebraically-central. Take any $k \in \text{Ker}(f)$. Now, for any $x \in A$ we calculate

$$[(x, x), (0, k)] = ([x, 0], [x, k]) = (0, [x, k]).$$

This being in $R(A \times_B A)$, $(\dagger)$ gives that $0 = [x, k]$. The same is of course true for any permutation of arguments, and so

$$0 = [x, k] = [k, x].$$

On the other hand, for any $x, y \in A$,

$$\langle (x, x), (y, y), (0, k) \rangle = (\langle x, y, 0 \rangle, \langle x, y, k \rangle) = (0, \langle x, y, k \rangle).$$

Similar to the above, $(\dagger)$ yields $0 = \langle x, y, k \rangle$, and the permutation of arguments yields

$$0 = \langle x, y, k \rangle = \langle x, k, y \rangle = \langle k, x, y \rangle.$$

Hence, $k \in Z(A)$, which finally gives $\text{Ker}(f) \subseteq Z(A)$.

□

As mentioned in the introduction, due to results from [4], this is enough to show the following:

Corollary 2.6. Let $f : A \rightarrow B$ be a regular epimorphism of K -algebras. f is classically-central if and only if it is categorically-central.

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