



VERY-WELL-BEHAVED EPIREFLECTIONS FOR CATEGORIES OF MODELS OF SKETCHES

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Résumé. Nous étudions les réflexions obtenues à partir d'autres adjonctions en factorisant l'unité de l'adjonction. L'objectif est de fournir des conditions garantissant la stabilité des unités de telles réflexions, et d'induire un système de factorisation *monotone-light*. En particulier, nous considérons les cas où l'adjonction est induite par une extension de Kan à gauche et à droite (respectivement). Dans la dernière section, le cas des extensions de Kan à droite à partir de catégories de préfaisceaux est restreint aux catégories de modèles de esquisses projectives, et une réflexion de catégories en préordres est donnée à titre d'exemple. Nous généralisons également aux 2- et n-catégories.

Abstract. We study reflections obtained from other adjunctions by factorizing the unit of the adjunction. The objective is to provide some conditions which ensure that such reflections have stable units, and induce a monotone-light factorization system. In particular, we consider the cases where the adjunction is induced by a left and right Kan extension (respectively). In last section, the case of right Kan extensions from categories of presheaves is restricted to categories of models of limit sketches, and a reflection of categories into preorders is given as an example. We also make a generalization to 2- and n-categories.

Keywords. Epireflection, stable units, prefactorization, monotone-light factorization, presheaves, simplicial set, Kan extensions, pseudo-filtered, cogen-

erating class, models of sketches, n -categories.

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1. Introduction

Let $(\mathcal{F}, \theta)$ be a pointed endofunctor on a category $\mathbb{C}$ with pullbacks, i.e., $\mathcal{F} : \mathbb{C} \rightarrow \mathbb{C}$ is an endofunctor and $\theta : 1_{\mathbb{C}} \rightarrow \mathcal{F}$ a natural transformation from the identity functor into $\mathcal{F}$ (see [6, §1.1]). If $(\mathcal{E}, \mathcal{M})$ is an (orthogonal) factorization system on $\mathbb{C}$ such that all the morphisms in $\mathcal{E}$ are epimorphisms and $\mathcal{E} = \mathcal{E}'$, the largest subclass of $\mathcal{E}$ stable under pullbacks, then there exists a *well-behaved* reflection $I : \mathbb{C} \rightarrow \mathbb{M}$ whose unit η is given by the componentwise decomposition of $\theta = \mu \circ \eta$ using the $(\mathcal{E}, \mathcal{M})$ -factorization. Well-behaved means that the induced reflection I has stable units, i.e., every unit morphism η_C belongs to $\mathcal{E}'_I$, the largest subclass of $\mathcal{E}_I$ which is stable under pullbacks, being $(\mathcal{E}_I, \mathcal{M}_I)$ the reflective factorization system associated to the reflection I (see [1]). Having stable units is a stronger condition than semi-left-exactness (see [2]), also called admissibility in categorical Galois theory.

The process described in the previous paragraph allows that, from an adjunction from a category with pullbacks and a stable factorization system (whose left-class consists of epimorphisms), it is induced a well-behaved reflection and the corresponding Galois theories.

It may happen that the induced well-behaved reflection is moreover *very-well-behaved*, meaning that there is a monotone-light factorization system $(\mathcal{E}'_I, \mathcal{M}_I^*)$ associated to it (see [1]). In order to be so, it only needs that, for every $C \in \mathbb{C}$, there is an effective descent morphism (see [5]) into C such that its domain is in the reflective subcategory.

In section 2 below it is presented a detailed study of the process outlined in the paragraphs above, followed by some generic examples and some results. These results were applied afterward to left and right Kan extensions for presheaves in sections 3 and 4 respectively.

In section 3, a new class of monotone-light factorizations were presented, deriving from adjunctions given by left Kan extensions for presheaves. A non-trivial example of such was displayed in an astonishingly simple case (cf. 3.5 below).

Then, in the rest of the paper, we focused in the more interesting context of right Kan extensions for presheaves, where there is an elegant characterization of the functors K in $Set^K \dashv Ran_K$ for which the existence of a very-well-behaved induced reflection is assured (see 4.5).

In the last section, Lemma 5.1 allows to move to the new level of well-behaved subreflections derived from the induced well-behaved reflections, from an adjunction of right Kan extensions for presheaves. The subreflection is obtained considering just the limit models of a sketch (cf. the beginning of section 5). The last part of the paper states that this process, ending in very-well-behaved subreflections of models of presheaves, encompasses the already known very-well-behaved reflections from n -categories into n -preorders (for every positive integer n ; cf. Examples 5.4, and subsections 5.1, 5.2).

It is noticeable that our main examples are about presheaves, $\mathbb{S} = Set$. In future work, we hope to enlarge this setting with new examples without this restriction, namely with $\mathbb{S}$ a regular or a (Barr) exact category, and to explore the algebraic setting (it is well known that a variety of universal algebras can be identified with a category of models of a sketch for presheaves).

Anyway, we believe that in this work a very detailed and precise theoretical framework was established, which makes us optimistic about future developments applying it.

Notice that all the basic concepts needed to read the following paper can be found in the bibliography, mostly in two items of it, namely [8] and [1].

2. Revision of General Results with Improvements

2.1 Derivable well-behaved epireflections

Let us first show that, a *well-behaved* epireflection is derivable from a monad in a category with pullbacks, such that certain of them are preserved by the endofunctor, provided there is a stable factorization system whose left class contains only epimorphisms (the needed conditions are in fact much weaker, as the reader shall discover; see Theorem 2.6 below).

Lemma 2.1.

Let $(\mathcal{F}, \theta)$ be a pointed endofunctor on $\mathbb{C}$ (pictured by $\mathbb{C} \xrightarrow{1_{\mathbb{C}}} \mathbb{C}$ with $\theta : 1_{\mathbb{C}} \rightarrow \mathcal{F}$).

If there is a prefactorization system $(\mathcal{E}, \mathcal{M})$ on $\mathbb{C}$ such that, for every $C \in \mathbb{C}$, each component factorizes as

$$\theta_C = \mu_C \circ \eta_C : C \rightarrow \mathcal{I}(C) \rightarrow \mathcal{F}(C),$$

with $\eta_C \in \mathcal{E}$ and $\mu_C \in \mathcal{M}$, then there is a unique pointed endofunctor $(\mathcal{I}, \eta)$ on $\mathbb{C}$ such that $\theta = \mu \cdot \eta$, where $\eta : 1_{\mathbb{C}} \rightarrow \mathcal{I}$ and $\mu : \mathcal{I} \rightarrow \mathcal{F}$ are the obvious natural transformations.

Proof. The proof follows trivially from the definition of a prefactorization system (cf. [1, §2.1]). Confer the diagram

$$\begin{array}{ccccc} C & \xrightarrow{\eta_C} & \mathcal{I}(C) & \xrightarrow{\mu_C} & \mathcal{F}(C) \\ f \downarrow & & \mathcal{I}f \downarrow & & \mathcal{F}f \downarrow \\ C' & \xrightarrow{\eta_{C'}} & \mathcal{I}(C') & \xrightarrow{\mu_{C'}} & \mathcal{F}(C'), \end{array}$$

where $\mathcal{I}f$ is the unique morphism making it commutative. $\square$

Lemma 2.2. Under the conditions of Lemma 2.1, if $\eta_C : C \rightarrow \mathcal{I}(C)$ is an epimorphism in $\mathbb{C}$ for every $C \in \mathbb{C}$, then the pointed endofunctor $(\mathcal{I}, \eta)$ on $\mathbb{C}$ is well-pointed, i.e., $\mathcal{I}\eta = \eta\mathcal{I}$.

Proof. η is an epimorphism in the functor category $\mathbb{C}^{\mathbb{C}}$ and

$$(\mathcal{I}\eta) \cdot \eta = \eta \circ \eta = (\eta\mathcal{I}) \cdot \eta$$

$\square$

In the following Proposition 2.3, a statement in [6, §1.1] is proved in detail.

Proposition 2.3. An endofunctor $(\mathcal{I}, \eta)$ on a category $\mathbb{C}$ is idempotent (i.e., it is well-pointed and, in addition, $\eta\mathcal{I}$ is an isomorphism) if and only if $\text{Fix}(\mathcal{I}, \eta)$ is a reflective full replete subcategory of $\mathbb{C}$ with reflection η , where $\text{Fix}(\mathcal{I}, \eta)$ is determined by all the objects $M \in \mathbb{C}$ for which $\eta_M : M \rightarrow \mathcal{I}(M)$ is an isomorphism.

Proof. $\text{Fix}(\mathcal{I}, \eta)$ is a replete subcategory of $\mathbb{C}$, since if $k : M \rightarrow N$ is an isomorphism from an object $M \in \text{Fix}(\mathcal{I}, \eta)$, then $\eta_N = \mathcal{I}k \circ \eta_M \circ k^{-1}$ is also an isomorphism, assuring that $N \in \text{Fix}(\mathcal{I}, \eta)$.

($\Rightarrow$ (only if))

If $(\mathcal{I}, \eta)$ is idempotent, then any image $\mathcal{I}(C)$ belongs to $Fix(\mathcal{I}, \eta)$, $C \in \mathbb{C}$ ($\eta\mathcal{I}$ is an isomorphism by hypothesis). Name $I = \mathcal{I}|_{Fix(\mathcal{I}, \eta)}$ the co-restriction of the functor $\mathcal{I}$ to $Fix(\mathcal{I}, \eta)$, and H the full inclusion of $Fix(\mathcal{I}, \eta)$ into $\mathbb{C}$. It is going to be shown that I is left adjoint to H , with unit η .

Let $f : C \rightarrow M$ be any morphism whose codomain M belongs to $Fix(\mathcal{I}, \eta)$. Then, $\eta_M \circ f = \mathcal{I}f \circ \eta_C \Leftrightarrow f = (\eta_M^{-1} \circ \mathcal{I}f) \circ \eta_C$. It remains to check that $f' = \eta_M^{-1} \circ \mathcal{I}f$ is the unique morphism such that $f = f' \circ \eta_C$:

$$\begin{aligned} f'' \circ \eta_C &= f' \circ \eta_C \Rightarrow \mathcal{I}(f'') \circ \mathcal{I}(\eta_C) = \mathcal{I}(f') \circ \mathcal{I}(\eta_C) \\ \Rightarrow \mathcal{I}(f'') \circ \eta_{\mathcal{I}(C)} &= \mathcal{I}(f') \circ \eta_{\mathcal{I}(C)} \quad (\mathcal{I}\eta = \eta\mathcal{I}, \text{ by hypothesis}) \\ \Rightarrow \eta_M \circ f'' &= \eta_M \circ f' \Rightarrow f'' = f' \quad (M \in Fix(\mathcal{I}, \eta)). \end{aligned}$$

($\Leftarrow$ (if))

If $\mathcal{I}(C)$ is the reflection of C , then $\mathcal{I}(C) \in Fix(\mathcal{I}, \eta)$ and $\eta_{\mathcal{I}(C)}$ is an isomorphism. Hence, $\eta\mathcal{I}$ is an isomorphism. Then, as $\eta\mathcal{I} \cdot \eta = \mathcal{I}\eta \cdot \eta$ and η is the unit of the reflection, it follows that $\eta\mathcal{I} = \mathcal{I}\eta$. $\square$

Theorem 2.4. *Under the conditions of the previous Lemmas 2.1 and 2.2, if $\mathcal{F}$ is a monad in which the unit is $\theta : 1_{\mathbb{C}} \rightarrow \mathcal{F}$, then $(\mathcal{I}, \eta)$ is idempotent, i.e., $\eta\mathcal{I}$ is an isomorphism (and $(\mathcal{I}, \eta)$ is well-pointed: $\mathcal{I}\eta = \eta\mathcal{I}$).*

More specifically, under the conditions of Lemma 2.2, if $\mathcal{F} = GF$, with $G \vdash F : \mathbb{C} \rightarrow \mathbb{X}$ being an adjunction with unit $\theta : 1_{\mathbb{C}} \rightarrow GF$, then there is a reflection $H \vdash I : \mathbb{C} \rightarrow \mathbb{M}$, where the right adjoint H is the full inclusion of $\mathbb{M} = Fix(HI, \eta)$. It also follows that $F\eta$ is an isomorphism.

Notice that $\mathbb{M} = Fix(\mathcal{I}, \eta)$ is equal to $\mathcal{M}(F)$, the full replete subcategory of $\mathbb{C}$ determined by those objects M such that $\theta_M : M \rightarrow \mathcal{F}(M)$ belongs to the right-class $\mathcal{M}$ of the prefactorization system on $\mathbb{C}$.

Proof. Consider the diagram

$$\begin{array}{ccccc} C & \xrightarrow{\eta_C} & \mathcal{I}(C) & \xrightarrow{\mu_C} & GF(C) \\ & & & \searrow f & \downarrow G(f') \\ & & & & G(X) \end{array} .$$

By the universality of the unit morphism $\theta_C = \mu_C \circ \eta_C$, there exists a unique morphism $f' : F(C) \rightarrow X$ such that $G(f') \circ \theta_C = f \circ \eta_C$, for every

$f : \mathcal{I}(C) \rightarrow G(X)$. Then $G(f') \circ \mu_C = f$ because η_C is an epimorphism. Hence, $\mu_C (\in \mathcal{M})$ is also a universal arrow from $\mathcal{I}(C)$ to G , and so it must be isomorphic to $\theta_{\mathcal{I}(C)} = \mu_{\mathcal{I}(C)} \circ \eta_{\mathcal{I}(C)}$ which implies that $\eta_{\mathcal{I}(C)} (\in \mathcal{E})$ is an isomorphism, by the properties of a prefactorization system $(\mathcal{E}, \mathcal{M})$ (cf. [1, Propositions 2.2(c) and 2.4]).

As F is a left-adjoint and every $\eta_C : C \rightarrow \mathcal{I}(C)$ is an epimorphism in $\mathbb{C}$, then $F\eta$ is an epimorphism. It is also split monic since $F = \varepsilon F \cdot F\theta = (\varepsilon F \cdot F\mu) \cdot F\eta$, with ε the counit of the adjunction $G \vdash F$. Therefore, $F\eta$ is an isomorphism. $\square$

Remark 2.5. Under the conditions of Theorem 2.4 and using the terminology in its statement, it is easy to conclude that $\eta_{G(X)}$ is an isomorphism for all $X \in \mathbb{X}$, since $G\varepsilon_X \circ \theta_{G(X)} = (G\varepsilon_X \circ \mu_{G(X)}) \circ \eta_{G(X)} = 1_{G(X)}$ and η_C is an epimorphism for all $C \in \mathbb{C}$, with ε the counit of the adjunction $G \vdash F$. Hence, the restriction $G_{\mathbb{M}} : \mathbb{X} \rightarrow \mathbb{M}$ of the right adjoint G to the codomain $\mathbb{M}$ is well defined. The left adjoint of $G_{\mathbb{M}}$ is obviously the restriction $F_{\mathbb{M}} : \mathbb{M} \rightarrow \mathbb{X}$ of F to $\mathbb{M}$. As $G = H \circ G_{\mathbb{M}}$, then $F \cong F_{\mathbb{M}} \circ I$. The unit of the adjunction $F_{\mathbb{M}} \dashv G_{\mathbb{M}}$ is exactly the restriction of μ to $\mathbb{M}$, provided one chooses $\eta_M = 1_M$ in every factorization $\theta_M = \mu_M \circ \eta_M$, for every $M \in \mathbb{M}$, which is equivalent to the fact that the identity $1_{\mathbb{M}}$ is the counit of the induced reflection ($I(M) = M$, for every $M \in \mathbb{M}$).

Theorem 2.6. *Under the conditions of Theorem 2.4, for a category $\mathbb{C}$ with pullbacks, the replete full reflection $I \dashv H : \mathcal{M}(F) \rightarrow \mathbb{C}$ has stable units if the following two conditions hold:*

(a) $\mathcal{F} = GF$ preserves the pullback squares of the form

$$\begin{array}{ccc} A \times_{\mathcal{I}(C)} C & \longrightarrow & C \\ g^*(\eta_C) \downarrow & & \downarrow \eta_C \\ A & \xrightarrow{g} & \mathcal{I}(C); \end{array}$$

(b) $\eta_C \in \mathcal{E}'$ for every $C \in \mathbb{C}$, where $\mathcal{E}'$ stands for the largest subclass of $\mathcal{E}$ which is closed under pullbacks.

Notice also that $I \dashv H : \mathcal{M}(F) \rightarrow \mathbb{C}$ does have stable units only if the condition (a) above holds, for a category $\mathbb{C}$ with pullbacks and under the

conditions of Theorem 2.4.

Proof. The final remark in the statement is going to be proved first. If the reflection $I \dashv H$ has stable units, then, by definition, the image by $\mathcal{I}$ of the pullback diagram in the statement is also a pullback diagram, meaning that both $\mathcal{I}\eta_C$ and $\mathcal{I}g^*(\eta_C)$ are isomorphisms (the former is so since $\mathcal{I}\eta = \eta\mathcal{I}$ is an isomorphism). Then, the image by $\mathcal{F}\mathcal{I}$ of the pullback diagram in the statement is necessarily a pullback diagram. Finally, $\mathcal{F}$ preserves the pullback diagram in the statement, since $F\eta$ being an isomorphism implies $\mathcal{F} \cong \mathcal{F}\mathcal{I}$.

For the if part of the theorem, consider the following commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{\eta_P} & \mathcal{I}(P) & \xrightarrow{\mu_P} & \mathcal{F}(P) \\ \downarrow g^*(\eta_C) & & \downarrow \mathcal{I}g^*(\eta_C) & & \downarrow \mathcal{F}g^*(\eta_C) \\ A & \xrightarrow{\eta_A} & \mathcal{I}(A) & \xrightarrow{\mu_A} & \mathcal{F}(A), \end{array}$$

where $P = A \times_{\mathcal{I}(C)} C$. We need to show that $\mathcal{I}g^*(\eta_C)$ is an isomorphism, or equivalently that $\mathcal{I}g^*(\eta_C) \in \mathcal{E} \cap \mathcal{M}$, using the properties of prefactorization systems (cf. [1, Proposition 2.4]): as $\eta_C \in \mathcal{E}'$, then $\eta_A \circ g^*(\eta_C) = \mathcal{I}g^*(\eta_C) \circ \eta_P \in \mathcal{E}$, implying that $\mathcal{I}g^*(\eta_C) \in \mathcal{E}$; as $\mathcal{F}g^*(\eta_C)$ is an isomorphism (by hypothesis (a), since $F\eta$ is an isomorphism) then $\mu_A \circ \mathcal{I}g^*(\eta_C) \cong \mu_P \in \mathcal{M}$, implying that $\mathcal{I}g^*(\eta_C) \in \mathcal{M}$ ($\mu_A \in \mathcal{M}$). $\square$

Remark 2.7. We take the opportunity here to report a mistake in our paper [16]. Theorem 2.4 in [16] corresponds to Theorem 2.6 just above, but in the latter the two conditions (a) and (b) are now only sufficient for the derived reflection to have stable units. Fortunately, this mistake had no impact on the other results of [16], since only the sufficiency was used.

Examples 2.8. Consider any adjunction $G \vdash F : \mathbb{C} \rightarrow \mathbb{X}$ with unit $\theta : 1_{\mathbb{C}} \rightarrow GF$, in which $\mathbb{C}$ is a regular category (i.e., it has pullbacks, it has coequalizers of kernel pairs, and the class of its regular epimorphisms is closed under pullbacks) and the left adjoint F preserves pullbacks. Then, there is a reflection $H \vdash I : \mathbb{C} \rightarrow \text{Mono}(F)$ with stable units, where $\text{Mono}(F)$ is a full (replete) subcategory of $\mathbb{C}$ determined by the objects $M \in \mathbb{C}$ for which the unit morphism $\theta_M : M \rightarrow GF(M)$ is a monomorphism. The

unit $\eta : 1_{\mathbb{C}} \rightarrow HI$ consists componentwise of regular epimorphisms of $\mathbb{C}$; in fact, $(\mathcal{E}, \mathcal{M}) = (\text{Regular Epimorphisms}, \text{Monomorphisms})$ is an (orthogonal) factorization system on $\mathbb{C}$.

2.2 Derivable very-well-behaved epireflections

Theorem 2.9. *Under the conditions of Theorem 2.4, if the derived reflection $H \vdash I : \mathbb{C} \rightarrow \mathcal{M}(F)$ has stable units, and for every object $C \in \mathbb{C}$ there is an effective descent morphism $p : M \rightarrow C$ (in $\mathbb{C}$) with domain in $\mathcal{M}(F)$, then there is a monotone-light factorization system $(\mathcal{E}_I^l, \mathcal{M}_I^*)$ on $\mathbb{C}$, obtained by simultaneously stabilizing and localizing the reflective factorization system $(\mathcal{E}_I, \mathcal{M}_I)$.*

Proof. This small result (but crucial in this work) is included in Corollary 6.2 in [12], which follows from the main result of [1]. $\square$

Examples 2.10. Under the conditions of Examples 2.8, let $\mathbb{C}$ be not only regular, but furthermore (Barr) exact, which is known to imply that in $\mathbb{C}$ the effective descent morphisms are the regular epimorphisms (cf. [5]). Suppose also that $\mathbb{C}$ is cocomplete, and that there is a full subcategory $\mathbb{N}$ of $\mathbb{C}$ which is dense in $\mathbb{C}$ (i.e., every $C \in \mathbb{C}$ is a colimit of objects of $\mathbb{N}$) and closed in $\text{Mono}(F)(= \mathcal{M}(F))$ for coproducts in $\mathbb{C}$ (hence, the canonical presentation of every $C \in \mathbb{C}$ as a colimit gives $p : M \rightarrow C$, with $M = \coprod_{i \in I} N_i \in \text{Mono}(F)$ and p a regular epimorphism, that is, an effective descent morphism; cf. [8, §V.2]).

Then, by Theorems 2.6 and 2.9, it follows that the induced reflection, from the left adjoint functor F preserving pullbacks, produces a monotone-light factorization system on $\mathbb{C}$.

Remark that, by next Lemma 2.11(b), instead of demanding that the dense subcategory $\mathbb{N}$ is closed in $\text{Mono}(F)$ for coproducts, one could alternatively ask $\mathbb{N}$ to be a dense subcategory of $\mathbb{C}$ closed under coproducts and such that the restriction of the left-adjoint F to $\mathbb{N}$ is faithful.

A class S of objects of a category $\mathbb{C}$ is a generating class if, for every two distinct parallel morphisms $f, f' \in \mathbb{C}$, there exists an object $N \in S$ and a morphism g with domain N such that $f \circ g \neq f' \circ g$.

Notice that the objects of a full subcategory $\mathbb{N}$ which is dense in $\mathbb{C}$ are obviously a generating class for $\mathbb{C}$.

The following Lemma 2.11 is needed in the proof of next Theorems 2.13 and 2.14, which give conditions under which an adjunction from a category of presheaves has *enough* effective descent morphisms, in the sense of Theorem 2.9.

Lemma 2.11. *Consider any adjunction $G \vdash F : \mathbb{C} \rightarrow \mathbb{X}$ with unit $\theta : 1_{\mathbb{C}} \rightarrow GF$.*

Suppose there is a full subcategory $\mathbb{N}$ of $\mathbb{C}$ whose class of objects $ob(\mathbb{N})$ is a generating class for $\mathbb{C}$. Then:

- (a) *for any object $C \in \mathbb{C}$, the unit morphism $\theta_C : C \rightarrow GF(C)$ is a monomorphism if and only if, for every object $N \in \mathbb{N}$, the restrictions $F_{N,C} : \mathbb{C}(N, C) \rightarrow \mathbb{X}(F(N), F(C))$ of the functor F are all injective;*
- (b) *[10, Lemma 4.2] each unit morphism $\theta_N : N \rightarrow GF(N)$ is a monomorphism, $N \in \mathbb{N}$, if and only if the restriction of the left -adjoint F to $\mathbb{N}$ is faithful.*

Proof. (a)

$$\begin{aligned} (\Rightarrow) Ff = Ff' : F(N) \rightarrow F(C) &\Rightarrow GFf \circ \theta_N = GFf' \circ \theta_N \\ &\Rightarrow \theta_C \circ f = \theta_C \circ f' \text{ (}\theta \text{ is a natural transformation)} \\ &\Rightarrow f = f' \text{ (}\theta_C \text{ is a monomorphism by hypothesis)} \end{aligned}$$

(in fact, it was proved that $F_{N,C}$ is injective for any $N \in \mathbb{C}$, provided θ_C is a monomorphism, and not only for $N \in \mathbb{N}$).

($\Leftarrow$) Suppose, by *reductio ad absurdum*, that there are $g \neq g' : C' \rightarrow C$ such that $\theta_C \circ g = \theta_C \circ g'$. Being $ob(\mathbb{N})$ a generating class for $\mathbb{C}$, there exists $\tau : N \rightarrow C'$ with $N \in \mathbb{N}$, such that $g \circ \tau \neq g' \circ \tau$. Then, $\theta_C \circ g \circ \tau = \theta_C \circ g' \circ \tau$ implies that $F(g \circ \tau) = F(g' \circ \tau)$, because θ_N is a unit morphism, and so $g \circ \tau = g' \circ \tau$ by hypothesis, which is a contradiction.

(b) The proof of this item follows immediately from item (a) by just considering all $C \in \mathbb{N}$. □

The following general result of Lemma 2.12 is needed in the proof of the consecutive Theorem 2.13.

Lemma 2.12. *Consider an adjunction $G \vdash F : \mathbb{C} \rightarrow \mathbb{X}$ with unit $\theta : 1_{\mathbb{C}} \rightarrow GF$. Suppose that, for the family $(T_i)_{i \in I}$ of objects of $\mathbb{C}$, there exists its*

coproduct $\coprod_{i \in I} T_i$ in $\mathbb{C}$. Then,

$$\theta_{\coprod_{i \in I} T_i} = \coprod_{i \in I} \theta_{T_i},$$

provided the right-adjoint G preserves the coproduct $\coprod_{i \in I} F(T_i)$ in $\mathbb{X}$.

Proof. Let $\iota_{T_i} : T_i \rightarrow \coprod_{i \in I} T_i$, $\iota_{F(T_i)} = F\iota_{T_i} : F(T_i) \rightarrow \coprod_{i \in I} F(T_i) = F(\coprod_{i \in I} T_i)$ and $\iota_{GF(T_i)} = G\iota_{F(T_i)} = GF\iota_{T_i} : GF(T_i) \rightarrow \coprod_{i \in I} GF(T_i) = G\coprod_{i \in I} F(T_i) = GF(\coprod_{i \in I} T_i)$ denote the three coproduct injections in question (recall that F preserves colimits, because it is a left adjoint).

Since θ is a natural transformation,

$$\theta_{\coprod_{i \in I} T_i} \circ \iota_{T_i} = GF(\iota_{T_i}) \circ \theta_{T_i}.$$

By definition of $\coprod_{i \in I} \theta_{T_i}$,

$$\coprod_{i \in I} \theta_{T_i} \circ \iota_{T_i} = \iota_{GF(T_i)} \circ \theta_{T_i}.$$

Hence, as $GF(\iota_{T_i}) = \iota_{GF(T_i)}$, $\theta_{\coprod_{i \in I} T_i} \circ \iota_{T_i} = \coprod_{i \in I} \theta_{T_i} \circ \iota_{T_i}$ for all $i \in I$, implying that $\theta_{\coprod_{i \in I} T_i} = \coprod_{i \in I} \theta_{T_i}$ by the definition of a coproduct $\coprod_{i \in I} T_i$. $\square$

It is convenient now to recall that, in a category of presheaves $\text{Set}^{\mathbb{D}^{op}}$, the representable functors $\mathbb{D}(-, D)$ constitute a generating class, $D \in \mathbb{D}$ (cf. [8, §III.7]).

Theorem 2.13. [10, See Remark 6.1] Consider any adjunction $G \vdash F : \mathbb{C} \rightarrow \mathbb{X}$ with unit $\theta : 1_{\mathbb{C}} \rightarrow GF$, where $\mathbb{C} = \text{Set}^{\mathbb{D}^{op}}$ is a category of presheaves: if

- $F \circ y$ is faithful
(where $y : \mathbb{D} \rightarrow \text{Set}^{\mathbb{D}^{op}}$ is the Yoneda embedding) and
- G preserves coproducts $\coprod_{i \in I} F(N_i)$,
where each $N_i = \mathbb{D}(-, D_i)$ belongs to the image of y ,

then

the unit morphism $\theta_{\coprod_{i \in I} \mathbb{D}(-, D_i)}$ is a monomorphism in $\mathbb{C} = \text{Set}^{\mathbb{D}^{op}}$, for each family $(D_i)_{i \in I}$ of objects of $\mathbb{D}$.

Proof. Since G preserves coproducts $\coprod_{i \in I} F(N_i)$, for each family $(N_i)_{i \in I}$ of objects as in the statement, $\theta_{\coprod_{i \in I} N_i} = \coprod_{i \in I} \theta_{N_i}$ (cf. Lemma 2.12). Hence, $\theta_{\coprod_{i \in I} \mathbb{D}(-, D_i)}$ is a monomorphism in $\mathbb{C} = \text{Set}^{\mathbb{D}^{op}}$, for each family $(D_i)_{i \in I}$ of objects of $\mathbb{D}$, by Lemma 2.11(b) and the fact that a monomorphism in $\text{Set}^{\mathbb{D}^{op}}$ is a componentwise injection. $\square$

The previous Theorem 2.13 will be helpful in section 3. The following Corollary 2.15 of Theorem 2.14 is needed in section 4.

Theorem 2.14. *Consider any adjunction $G \vdash F : \mathbb{C} \rightarrow \mathbb{X}$ with unit $\theta : 1_{\mathbb{C}} \rightarrow GF$, where $\mathbb{C} = \text{Set}^{\mathbb{D}^{op}}$ is a category of presheaves: if*

- $F \circ y$ is faithful and injective on objects
(where $y : \mathbb{D} \rightarrow \text{Set}^{\mathbb{D}^{op}}$ is the Yoneda embedding),
- the coproduct injections in $\mathbb{X}$ of the form $F(\mathbb{D}(-, D_j)) \rightarrow \coprod_{i \in I} F(\mathbb{D}(-, D_i))$

are monomorphisms,

- the equality $\iota_j \circ f = \iota_{j'} \circ f'$ implies $\iota_j = \iota_{j'}$,

for every two coproduct injections

$\iota_j : F(\mathbb{D}(-, D_j)) \rightarrow \coprod_{i \in I} F(\mathbb{D}(-, D_i))$ and $\iota_{j'} : F(\mathbb{D}(-, D_{j'})) \rightarrow \coprod_{i \in I} F(\mathbb{D}(-, D_i))$ in $\mathbb{X}$,

and any pair of morphisms f, f' with common domain $F(\mathbb{D}(-, D))$, for some $D \in \mathbb{D}$,

then

the unit morphism $\theta_{\coprod_{i \in I} \mathbb{D}(-, D_i)}$ is a monomorphism in $\mathbb{C} = \text{Set}^{\mathbb{D}^{op}}$, for each family $(D_i)_{i \in I}$ of objects of $\mathbb{D}$.

Proof. The representable functors $\mathbb{D}(-, D)$ in $\text{Set}^{\mathbb{D}^{op}}$ are known to constitute a dense full subcategory of $\text{Set}^{\mathbb{D}^{op}}$ (cf. [8, Theorem 1 in §III.7]), therefore a generating class for $\text{Set}^{\mathbb{D}^{op}}$. By Lemma 2.11(a), we have just to show that, for every family $(D_i)_{i \in I}$ of objects in $\mathbb{D}$, $F_{\mathbb{D}(-, D), \coprod_{i \in I} \mathbb{D}(-, D_i)} :$

$\text{Set}^{\mathbb{D}^{op}}(\mathbb{D}(-, D), \coprod_{i \in I} \mathbb{D}(-, D_i)) \rightarrow \mathbb{X}(F(\mathbb{D}(-, D)), F(\coprod_{i \in I} \mathbb{D}(-, D_i)))$ is an injection, for every $D \in \mathbb{D}$.

It is easy to show, using the Yoneda Lemma, that for any $\alpha : \mathbb{D}(-, D) \rightarrow \coprod_{i \in I} \mathbb{D}(-, D_i)$ there exists $\alpha_j : \mathbb{D}(-, D) \rightarrow \mathbb{D}(-, D_j)$, with $j \in I$, such that $\delta_j \circ \alpha_j = \alpha$, where $\delta_j : \mathbb{D}(-, D_j) \rightarrow \coprod_{i \in I} \mathbb{D}(-, D_i)$ is a coproduct injection, $j \in I$ (α_j is determined by $\alpha_D(1_D) : D \rightarrow D_j$, i.e., $\alpha_{j,D}(1_D) = \alpha_D(1_D)$).

Let $\alpha' : \mathbb{D}(-, D) \rightarrow \coprod_{i \in I} \mathbb{D}(-, D_i)$ be another morphism in $\text{Set}^{\mathbb{D}^{op}}$ such that $F\alpha' = F\alpha$; then, there exists $\alpha'_{j'} : \mathbb{D}(-, D) \rightarrow \mathbb{D}(-, D_{j'})$ such that $\delta_{j'} \circ \alpha'_{j'} = \alpha'$, where $\delta_{j'} : \mathbb{D}(-, D_{j'}) \rightarrow \coprod_{i \in I} \mathbb{D}(-, D_i)$ is a coproduct injection ($j' \in I$); hence, $F\delta_j \circ F\alpha_j = F\delta_{j'} \circ F\alpha'_{j'}$. Notice that $\iota_j = F\delta_j$ and $\iota_{j'} = F\delta_{j'}$ are coproduct injections in $\mathbb{X}$ (since F is a left-adjoint). By the hypotheses in the statement, $\iota_j = \iota_{j'}$, which implies $\delta_j = \delta_{j'}$ ($F\delta_j = F\delta_{j'} \Rightarrow F(\mathbb{D}(-, D_j)) = F(\mathbb{D}(-, D_{j'})) \Rightarrow \mathbb{D}(-, D_j) = \mathbb{D}(-, D_{j'})$, because Fy is injective on objects) and $F\alpha_j = F\alpha'_{j'}$ (ι_j is a monomorphism by hypothesis). Then, $\alpha_j = \alpha_{j'}$ because Fy is faithful (it was concluded before that α_j and $\alpha'_{j'}$ are parallel morphisms in the dense subcategory of representable functors). Finally, $\alpha = \delta_j \circ \alpha_j = \delta_{j'} \circ \alpha'_{j'} = \alpha'$. $\square$

Corollary 2.15. [10, See footnote in Remark 6.1] *If both $\mathbb{C} = \text{Set}^{\mathbb{D}^{op}}$ and $\mathbb{X} = \text{Set}^{\mathbb{B}}$ are categories of presheaves, and $F \circ y$ is faithful and injective on objects (where $y : \mathbb{D} \rightarrow \text{Set}^{\mathbb{D}^{op}}$ is the Yoneda embedding), then the unit morphism $\theta_{\coprod_{i \in I} \mathbb{D}(-, D_i)}$ is a monomorphism in $\mathbb{C} = \text{Set}^{\mathbb{D}^{op}}$, for each family $(D_i)_{i \in I}$ of objects of $\mathbb{D}$.*

Proof. First, it is trivial that the coproduct injections are monomorphisms in $\text{Set}^{\mathbb{B}}$.

According to Theorem 2.14, if $\iota_j \circ f = \iota_{j'} \circ f'$ implies that $\iota_j = \iota_{j'}$ in $\text{Set}^{\mathbb{B}}$ (where $\iota_j : X_j \rightarrow \coprod_{i \in I} X_i$ and $\iota_{j'} : X_{j'} \rightarrow \coprod_{i \in I} X_i$ are coproduct injections in $\text{Set}^{\mathbb{B}}$, for any family $(X_i)_{i \in I}$ of objects in $\text{Set}^{\mathbb{B}}$) then the statement would have been proved:

in fact, for every $B \in \mathbb{B}$, $\iota_{j,B} \circ f_B = \iota_{j',B} \circ f'_B$ implies that $X_j(B) = X_{j'}(B)$ since $(\coprod_{i \in I} X_i)(B) = \coprod_{i \in I} X_i(B)$ and $\iota_{j,B}, \iota_{j',B}$ are inclusions in Set ; therefore, $\iota_{j,B} = \iota_{j',B}$ for every $B \in \mathbb{B}$, that is, $\iota_j = \iota_{j'}$. $\square$

2.3 Uplifting factorization systems to functor categories

The last result in this section 2 allows us to start with a factorization of the morphisms in a category $\mathbb{S}$, and then raise it to the factorization of the natural transformations in the functor category $\mathbb{S}^{\mathbb{I}}$.

Lemma 2.16. *Let $\mathcal{S}$ be a class of morphisms in the category $\mathbb{S}$. Define $\mathcal{S}^{\mathbb{I}}$ as the class of all morphisms in the functor category $\mathbb{S}^{\mathbb{I}}$ whose components are in $\mathcal{S}$ ($\alpha = (\alpha_i)_{i \in \mathbb{I}} \in \mathcal{S}^{\mathbb{I}}$ if and only if α_i is in $\mathcal{S}$ for all $i \in \mathbb{I}$).*

(a) If $(\mathcal{E}, \mathcal{M})$ is a prefactorization system on $\mathbb{S}$, then there is a prefactorization system $(\mathcal{F}, \mathcal{N})$ on $\mathbb{S}^\mathbb{I}$ such that $\mathcal{E}^\mathbb{I} \subseteq \mathcal{F}$ and $\mathcal{M}^\mathbb{I} \subseteq \mathcal{N}$.

(b) If $(\mathcal{E}, \mathcal{M})$ is a factorization system on $\mathbb{S}$, then $(\mathcal{E}^\mathbb{I}, \mathcal{M}^\mathbb{I})$ is a factorization system on $\mathbb{S}^\mathbb{I}$.

Proof. (a) It has to be proved that, for any commutative square $v \circ e = m \circ u$ in $\mathbb{S}^\mathbb{I}$, with $e : A \rightarrow B \in \mathcal{E}^\mathbb{I}$ and $m : C \rightarrow D \in \mathcal{M}^\mathbb{I}$, there is one and only one morphism $w : B \rightarrow C$ such that $w \circ e = u$ and $m \circ w = v$.

If such a morphism w exists then it must be the unique $w = (w_i)_{i \in \mathbb{I}}$ such that $w_i \circ e_i = u_i$ and $m_i \circ w_i = v_i$, for every $i \in \mathbb{I}$. The question is if $(w_i)_{i \in \mathbb{I}}$ is a natural transformation.

Let $f : i \rightarrow j$ be a morphism in $\mathbb{I}$. It has to be shown that $Cf \circ w_i = w_j \circ Bf$, or equivalently (since $e_i \downarrow m_j$), $Cf \circ w_i \circ e_i = Cf \circ u_i = w_j \circ Bf \circ e_i$ and $m_j \circ Cf \circ w_i = v_j \circ Bf = m_j \circ w_j \circ Bf$:

$Cf \circ (w_i \circ e_i) = Cf \circ u_i$, $w_j \circ (Bf \circ e_i) = (w_j \circ e_j) \circ Af = u_j \circ Af = Cf \circ u_i$; $(m_j \circ Cf) \circ w_i = Df \circ (m_i \circ w_i) = Df \circ v_i = v_j \circ Bf$, $(m_j \circ w_j) \circ Bf = v_j \circ Bf$.

(b) Being $(\mathcal{E}, \mathcal{M})$ a factorization system on $\mathbb{S}$, it is obvious that both $\mathcal{E}^\mathbb{I}$ and $\mathcal{M}^\mathbb{I}$ contain the identities and are closed under composition with isomorphisms. By (a), in order that $(\mathcal{E}^\mathbb{I}, \mathcal{M}^\mathbb{I})$ is a factorization system on $\mathbb{S}^\mathbb{I}$, it remains to show that each morphism $\alpha : A \rightarrow B$ in $\mathbb{S}^\mathbb{I}$ factorises as $\alpha = m \circ e$, with $m \in \mathcal{M}^\mathbb{I}$ and $e \in \mathcal{E}^\mathbb{I}$ (cf. [1, §2.8]):

$$\begin{array}{ccccc} A(i) & \xrightarrow{e_i} & C(i) & \xrightarrow{m_i} & B(i) & m_i \circ e_i = \alpha_i \\ Af \downarrow & & Cf \downarrow & & Bf \downarrow & \\ A(j) & \xrightarrow{e_j} & C(j) & \xrightarrow{m_j} & B(j) & m_j \circ e_j = \alpha_j, \end{array}$$

where $C : \mathbb{I} \rightarrow \mathbb{S}$ being a functor follows from $(\mathcal{E}, \mathcal{M})$ being a (pre)factorization system. $\square$

3. The case of left Kan extensions for presheaves

Theorem 3.1. Consider a functor $K : \mathbb{B} \rightarrow \mathbb{A}$, such that $\mathbb{B}$ is a small category and $\mathbb{A}$ is a category with small hom-sets. Then, there is an adjunction $\text{Set}^K \vdash \text{Lan}_K : \text{Set}^\mathbb{B} \rightarrow \text{Set}^\mathbb{A}$ given by left Kan extensions.

If the left-adjoint Lan_K preserves pullbacks and $\text{Lan}_K \circ y$ is faithful (where $y : \mathbb{B}^{\text{op}} \rightarrow \text{Set}^\mathbb{B}$ is the Yoneda embedding), then there is an induced

reflection with stable units $H \vdash I : \text{Set}^{\mathbb{B}} \rightarrow \text{Mono}(\text{Lan}_K)$ and a monotone-light factorization $(\mathcal{E}'_I, \mathcal{M}^*_I)$ on $\text{Set}^{\mathbb{B}}$.

Proof. The statement is a special case of Examples 2.10, with $\mathbb{C} = \text{Set}^{\mathbb{B}}$, $\mathbb{X} = \text{Set}^{\mathbb{A}}$, $(\mathcal{E}, \mathcal{M}) = (\text{Epis}, \text{Monos})$ is the lifting given by Lemma 2.16(b) of the factorization system $(\text{Surjections}, \text{Injections})$ on Set .

In fact, the representable functors $\mathbb{B}(B, -)$ in $\text{Set}^{\mathbb{B}}$ constitute a dense subcategory of $\text{Set}^{\mathbb{B}}$, whose unit morphisms

$$\theta_{\mathbb{B}(B, -)} : \mathbb{B}(B, -) \rightarrow \text{Lan}_K(\mathbb{B}(B, -)) \circ K$$

are by assumption monomorphisms ($\Leftrightarrow \text{Lan}_K \circ y$ faithful) and their co-products are also in $\text{Mono}(\text{Lan}_K)$ by Theorem 2.13 (the right adjoint Set^K preserves colimits). $\square$

Remark 3.2. In Theorem 3.1 just above, instead of asking that Lan_K preserves all pullbacks, it could be only demanded the weaker condition that $\text{Set}^K \circ \text{Lan}_K$ preserves pullbacks (in fact, only the ones given in Theorem 2.6(a)).

Notice that, if $K = ! : \mathbb{J} \rightarrow \mathbf{1}$ is the functor whose codomain is the category $\mathbf{1}$ (one object and no non-identity morphisms) and the domain $\mathbb{J}$ is small, then the adjunction $\text{Set}^K \vdash \text{Lan}_K : \text{Set}^{\mathbb{J}} \rightarrow \text{Set}$ is $\Delta \vdash \text{Colim}$, that is, the right adjoint is the diagonal functor and the left adjoint takes colimits.

Recall, as well, that a category $\mathbb{J} = \coprod_{i \in I} \mathbb{J}_i$ is pseudo-filtered if its connected components $\mathbb{J}_i$ are filtered (cf. Exercise 2 in [8, §IX.2]).

Lemma 3.3. *Consider the adjunction $\Delta \vdash \text{Colim} : \text{Set}^{\mathbb{J}} \rightarrow \text{Set}$, with unit θ , from which derives the full reflective subcategory $\text{Mono}(\text{Colim})$ as in Theorem 2.4, using the factorization system*

$$(\text{Epis}, \text{Monos}) = (\text{Surjections}^{\mathbb{J}}, \text{Injections}^{\mathbb{J}})$$

(cf. Lemma 2.16(b)). *Let the category $\mathbb{J}$ be pseudo-filtered (cf. last paragraph just before this Lemma 3.3). Then, the two following statements are valid.*

- (a) *$M \in \text{Mono}(\text{Colim})$ if and only if Mu is injective, for every u in $\mathbb{J}$.*
- (b) *The functor $\text{Colim} : \text{Set}^{\mathbb{J}} \rightarrow \text{Set}$ preserves pullbacks.*

Proof. (a)

$(\Rightarrow) M \in \text{Mono}(\text{Colim}) \Leftrightarrow \forall_{j \in \mathbb{J}} \theta_{M,j} : M(j) \rightarrow \text{Colim}(M)$ is an injection, where $\theta_{M,j}$ is the j -component of the unit morphism of $M \in \text{Set}^{\mathbb{J}}$ (the colimit cocone of $M : \mathbb{J} \rightarrow \text{Set}$), then

$$\forall_{u:j \rightarrow j'} \forall_{x,x' \in M(j)} Mu(x) = Mu(x') \Rightarrow x = x'.$$

In fact, $\theta_{M,j}(x) = \theta_{M,j'} \circ Mu(x) = \theta_{M,j'} \circ Mu(x') = \theta_{M,j}(x')$, and so $x = x'$ because $\theta_{M,j}$ is an injection:

$$\begin{array}{ccc} & \text{Colim}(M) & \\ \theta_{M,j} \nearrow & & \nwarrow \theta_{M,j'} \\ M(j) & \xrightarrow{Mu} & M(j') \end{array} .$$

$(\Leftarrow) \text{Colim}(M) = \coprod_{j \in \mathbb{J}} M(j)/E$, where E is the equivalence relation which is about to be described. Let $x \in M(j)$ and $x' \in M(j')$, then xEx' if and only if there exists $u : j \rightarrow k$ and $u' : j' \rightarrow k$ such that $Mu(x) = Mu'(x')$ (cf. the proof of Theorem 1 in [8, §IX.2], where this result is stated for $\mathbb{J}$ filtered; the generalization to $\mathbb{J} = \coprod_{i \in I} \mathbb{J}_i$ pseudo-filtered is obvious, since the colimits are obtained as the disjoint union of the colimits of the filtered connected components $\mathbb{J}_i$).

Let $x, x' \in M(j)$ such that $\theta_{M,j}(x) = \theta_{M,j}(x')$, and therefore xEx' . Then, as $j = j'$, it follows that $\exists j \xrightarrow[u']{u} k \xrightarrow{w} k'$ such that $M(w \circ u)(x) = M(w \circ u')(x')$, which implies by the hypothesis that $x = x'$, since $w \circ u = w \circ u'$ (the existence of such $w : k \rightarrow k'$ is assured by $\mathbb{J}$ being pseudo-filtered).

(b) It is well known that finite limits commute with filtered colimits in Set . The statement is just a refinement of this fact (cf. Exercise 4 in [8, §IX.2]): pseudo-filtered colimits commute with pullbacks in Set ; it follows that if $\mathbb{J}$ is pseudo-filtered then Colim preserves pullbacks. $\square$

Theorem 3.4. *Under the conditions of Lemma 3.3, if every morphism in $\mathbb{J}$ is a monomorphism, then there is a monotone-light factorization system $(\mathcal{E}'_I, \mathcal{M}^*_I)$ on $\text{Set}^{\mathbb{J}}$.*

Proof. By Lemma 3.3(b), the left-adjoint $\text{Lan}_! = \text{Colim}$ preserves pullbacks. According to Theorem 3.1, there exists the required monotone-light factorization, provided any representable functor $\mathbb{J}(i, -)$ belongs to

$Mono(Colim)$. This is the case if, for every $u : j \rightarrow j'$ in $\mathbb{J}$, the image $\mathbb{J}(i, -)u : \mathbb{J}(i, -)(j) \rightarrow \mathbb{J}(i, -)(j')$ is always an injection, by Lemma 3.3(a). It is obviously so, since the functions (compose with u) $u_* : \mathbb{J}(i, j) \rightarrow \mathbb{J}(i, j')$ are injections if u is a monomorphism. $\square$

Examples 3.5. Any preorder $\mathbb{J}$, such that each of its connected components is a filtered category (also called filtered set or directed set), gives rise to a very-well-behaved induced reflection $Set^{\mathbb{J}} \rightarrow Mono(Colim)$.

For instance, consider the very simple special case of Theorem 3.4 with $\mathbb{J} = \mathbf{2}$, the ordinal number with two objects 0, 1 and only one non-identity morphism $u : 0 \rightarrow 1$. As $\mathbf{2}$ is a filtered preorder, the reflection $H \vdash I : Set^{\mathbf{2}} \rightarrow Mono(Colim)$ is *very-well-behaved*, that is, has stable units and a monotone-light factorization $(\mathcal{E}'_I, \mathcal{M}^*_I)$, obtained by simultaneous stabilization and localization of the classes in the reflective factorization system $(\mathcal{E}_I, \mathcal{M}_I)$. It is going to be shown that this particular monotone-light factorization $(\mathcal{E}'_I, \mathcal{M}^*_I)$ is non-trivial, that is, $\mathcal{E}'_I$ is strictly contained in $\mathcal{E}_I$ ($\Leftrightarrow \mathcal{M}^*_I$ strictly contains $\mathcal{M}_I$).

The unit morphism $\theta_M : M \rightarrow \Delta Colim(M) = \Delta M(1)$ consists in two functions $\theta_{M,0} = Mu$ and $\theta_{M,1} = id_{M(1)}$, for all $Mu = M(0) \rightarrow M(1)$. Hence, from the $(Surjections, Injections)$ -factorizations

$$\theta_{M,0} = \mu_{M,0} \circ \eta_{M,0}, \quad \theta_{M,1} = 1_{M(1)} = \mu_{M,1} \circ \eta_{M,1} = 1_{M(1)} \circ 1_{M(1)},$$

it is easy to conclude that the replete full subcategory $Mono(Colim)$ is determined by all injections (cf. Lemma 3.3). Concluding then that a morphism $\varphi = (\varphi_0, \varphi_1) : M \rightarrow N$ belongs to $\mathcal{E}_I$ if and only if $\varphi_1 : M(1) \rightarrow N(1)$ is a bijection and $(Nu \circ \varphi_0)(M(0)) = Nu(N(0))$ (where $Nu : N(0) \rightarrow N(1)$ and $\varphi_0 : M(0) \rightarrow N(0)$):

indeed, $\varphi = (\varphi_0, \varphi_1) \in \mathcal{E}_I$ if and only if $I\varphi = I(\varphi_0, \varphi_1) = (I(\varphi)_0, I(\varphi)_1)$ is an isomorphism (cf. [1, (3.2) in 3.1]), that is, both components are bijections. Noticing that $I(\varphi)_1 = \varphi_1$, and that φ_1 being a bijection implies that $I(\varphi)_0$ is an injection, it remains to characterize the surjection of $I(\varphi)_0$ in terms of the initial data: $I(\varphi)_0$ is a surjection iff $I(\varphi)_0 \circ \eta_{M,0}$ is a surjection iff $\eta_{N,0} \circ \varphi_0$ is a surjection iff $\eta_{N,0} \circ \varphi_0(M(0)) = I(N)(0)$ iff $Nu \circ \varphi_0(M(0)) = Nu(N(0))$ ($\cong I(N)(0)$).

Let $M = N$, $M(0) = \{x, y\}$, $M(1) = \{x\}$ and $\varphi_0(x) = x = \varphi_0(y)$. This particular morphism $\varphi = (\varphi_0, \varphi_1)$ belongs to $\mathcal{E}_I$ but does not belong to

its largest subclass closed under pullbacks $\mathcal{E}'_I$. Indeed, take $Qu : \{y\} \rightarrow \{x\}$, and $\psi = (\psi_0, \psi_1) : Q \rightarrow N$, such that ψ_0 and ψ_1 are the inclusions, then the pullback $\psi^*(\varphi)$ of φ along ψ does not belong to $\mathcal{E}_I$, since the empty set is the domain of $(\psi^*(\varphi))_0 : \emptyset \rightarrow \{y\}$.

4. The case of right Kan extensions for presheaves

The right Kan extensions adjunction $Ran_K \vdash Set^K : Set^{\mathbb{A}} \rightarrow Set^{\mathbb{B}}$ is more interesting for the purpose of obtaining *very-well-behaved* reflections, than the left Kan extensions case of the precedent section 3.

As Set^K is left-exact “a priori” (in fact, it preserves both limits and colimits), Corollary 2.15 (of Theorem 2.14; cf. also Examples 2.10) states that $H \vdash I : Set^{\mathbb{A}} \rightarrow Mono(Set^K)$ is *very-well-behaved* if $Set^K \circ y$ is faithful ($\Leftrightarrow \forall_{A \in \mathbb{A}} \mathbb{A}(A, -) \in Mono(Set^K)$) and injective on objects, where $y : \mathbb{A}^{op} \rightarrow Set^{\mathbb{A}}$ is the Yoneda embedding. But, for the case $Ran_K \vdash Set^K : Set^{\mathbb{A}} \rightarrow Set^{\mathbb{B}}$, the injectivity on objects of $Set^K \circ y$ is not needed, because coproduts preserve jointly monic morphisms in Set (see Definition 4.3 and Lemma 4.4).

Next Theorem 4.5 gives a useful characterization of such a case using the notion of *cogenerating class* (the dual of the notion of a generating class given two paragraphs before Lemma 2.11).

Lemma 4.1. *Consider an adjunction*

$$(\mathbb{S}^K, Ran_K, \theta, \varepsilon) : \mathbb{S}^{\mathbb{A}} \rightarrow \mathbb{S}^{\mathbb{B}},$$

given by right Kan extensions calculated as pointwise limits. For every $S \in \mathbb{S}^{\mathbb{A}}$, the right Kan extension $Ran_K(SK)$ is calculated for each $A \in \mathbb{A}$ as

$$Ran_K(SK)(A) = Lim((A \downarrow K) \xrightarrow{Q} \mathbb{B} \xrightarrow{SK} \mathbb{S}) = Lim_f SK(B),$$

$f \in (A \downarrow K)$, where Q is the projection of the comma category (cf. [8, §X.3]).

Then, the unit morphism θ_S is the unique morphism in $\mathbb{S}^{\mathbb{A}}$ such that

$$\lambda_f^{SK, A} \circ \theta_{S, A} = Sf,$$

for every $f \in (A \downarrow K)$, where $\lambda_f^{SK, A} : Ran_K(SK)(A) \rightarrow SK(B)$ is the morphism in the limiting cone of $Ran_K(SK)(A)$ associated with $f : A \rightarrow K(B)$.

Proof. Theorem 1 in [8, §X.3] gives the counit and proves the universal property for the adjunction in the statement. Finding the unit is just a matter of applying the universal property in the special case of an identity natural transformation, as in every adjunction. $\square$

Lemma 4.2. *Under the conditions in the previous Lemma 4.1, for each $S \in \mathbb{S}^{\mathbb{A}}$ and $A \in \mathbb{A}$, $\theta_{S,A} : S(A) \rightarrow \text{Ran}_K(SK)(A)$ is a monomorphism in $\mathbb{S}$ if and only if the morphisms in the set $\{Sf : S(A) \rightarrow SK(B) \mid f : A \rightarrow K(B), B \in \mathbb{B}\}$ are jointly monic in $\mathbb{S}$.*

Proof. The easy proof is displayed in [16, Lemma 4.1]. The immediately above Lemma 4.1 completes that proof, by justifying the nature of the unit θ assumed in [16, Lemma 4.1]. $\square$

Definition 4.3. *Consider, in a category $\mathbb{S}$ with coproducts, a family $(f_{ij} : s_i \rightarrow s_{ij})_{j \in J}$ of families of morphisms in $\mathbb{S}$ with the same domain s_i which are jointly monic, for each $i \in I$ (notice that J is fixed). It will be said that the coproducts preserve jointly monic morphisms in $\mathbb{S}$ if, for any such family of families, the family $(\coprod_{i \in I} f_{ij} : \coprod_{i \in I} s_i \rightarrow \coprod_{i \in I} s_{ij})_{j \in J}$ is also jointly monic, where $\coprod_{i \in I} f_{ij}$ is the morphism uniquely determined by the coproduct diagram associated to $(f_{ij})_{i \in I}$.*

As an example, it is trivial that coproducts preserve jointly monic morphisms in the category Set of sets.

Lemma 4.4. *Under the conditions of Lemmas 4.1, suppose that $\mathbb{S}$ has coproducts which preserve jointly monic morphisms (cf. previous Definition 4.3 just above).*

If $(S_i)_{i \in I}$ is a family of functors belonging to $\mathbb{S}^{\mathbb{A}}$, such that $\theta_{S_i} : S_i \rightarrow \text{Ran}_K(S_i K)$ is monic, for all $i \in I$, then the unit $\theta_{\coprod_{i \in I} S_i}$ of the coproduct is also monic.

Proof. We need to show that $\theta_{\coprod_{i \in I} S_i, A} : \coprod_{i \in I} S_i(A) \rightarrow \text{Ran}_K(\coprod_{i \in I} S_i K)(A)$ is a monomorphism for every $A \in \mathbb{A}$; or, equivalently, that the morphisms in the sets

$$\left\{ \coprod_{i \in I} S_i f : \coprod_{i \in I} S_i(A) \rightarrow \coprod_{i \in I} S_i K(B) \mid f : A \rightarrow K(B), B \in \mathbb{B} \right\}$$

are jointly monic for every $A \in \mathbb{A}$, according to Lemma 4.2.

Since $\theta_{S_i, A} : S_i(A) \rightarrow \text{Ran}_K(S_i K)(A)$ is a monomorphism by assumption, for all $i \in I$ and $A \in \mathbb{A}$, then all the sets $\{S_i f : S_i(A) \rightarrow S_i K(B) \mid f : A \rightarrow K(B), B \in \mathbb{B}\}$ are jointly monic, again by Lemma 4.2. Hence, as coproducts preserve jointly monic morphisms in $\mathbb{S}$, it follows that $\theta_{\coprod_{i \in I} S_i}$ is monic. $\square$

Theorem 4.5. *Let $K : \mathbb{B} \rightarrow \mathbb{A}$ be a functor from a small category $\mathbb{B}$ into a category $\mathbb{A}$ with small hom-sets. There is an adjunction $\text{Ran}_K \vdash \text{Set}^K : \text{Set}^{\mathbb{A}} \rightarrow \text{Set}^{\mathbb{B}}$ given by right Kan extensions calculated as pointwise limits. Then, factorizing its unit morphisms $\theta_S : S \rightarrow \text{Ran}_K(SK)$ componentwise with the surjections and injections, gives rise to the full reflection with stable units $H \vdash I : \text{Set}^{\mathbb{A}} \rightarrow \text{Mono}(\text{Set}^K)$.*

If $K(\text{ob}\mathbb{B})$ is a cogenerating class for $\mathbb{A}$, then $(\mathcal{E}'_I, \mathcal{M}^*_I)$ is a monotone-light factorization system on $\text{Set}^{\mathbb{A}}$ (where $K(\text{ob}\mathbb{B})$ is the image by K of every object in $\mathbb{B}$).

Proof. Consider two morphisms $f, g : A' \rightarrow A$ in $\mathbb{A}$, and their image $f^*, g^* : \mathbb{A}(A, K(-)) \rightarrow \mathbb{A}(A', K(-))$ by $\text{Set}^K \circ y$. These two morphisms f^* and g^* are distinct functors if and only if there exists $B \in \mathbb{B}$ and a morphism $h : A \rightarrow K(B)$ such that $f^*_B(h) = h \circ f \neq h \circ g = g^*_B(h)$ (see [10], just before Theorem 6.1 there, where this part of the proof was given).

In the last paragraph, it has been shown that $\text{Set}^K \circ y$ is faithful if and only if $K(\text{ob}\mathbb{B})$ is a cogenerating class for $\mathbb{A}$. It then follows, by Lemma 2.11(b) that all representable functors $\mathbb{A}(A, -)$ are in $\text{Mono}(\text{Set}^K)$, $A \in \mathbb{A}$, if and only if $K(\text{ob}\mathbb{B})$ is a cogenerating class for $\mathbb{A}$. Then, by the precedent Lemma 4.4, with $\mathbb{S} = \text{Set}$, all the coproducts $\coprod_{i \in I} \mathbb{A}(A_i, -)$ are in $\text{Mono}(\text{Set}^K)$. Hence, the conditions of Theorem 2.9 hold (cf. Examples 2.10), and so there is a monotone-light factorization system derived from $\text{Set}^K \dashv \text{Ran}_K$ if $K(\text{ob}\mathbb{B})$ is a cogenerating class for $\mathbb{A}$. $\square$

Examples 4.6. Consider the category Δ of non-empty finite ordinal numbers $[n]$ ($n \geq 0$), where the singular set $\{[0]\}$ is a generating class ($\Delta = \Delta^+$ as in [8, VII.5]: $[n]$ is the linear order whose objects are $0, 1, \dots, n$; the morphisms of Δ are the weakly monotone functions). Hence, $\{[0]\}$ is a cogenerating class in Δ^{op} . Let $K : \mathbb{B} \rightarrow \Delta^{op}$ be a functor (from a small category $\mathbb{B}$) such that $[0]$ is in the image $K(\text{ob}\mathbb{B})$ of the objects of $\mathbb{B}$. Then, $H \vdash I : \text{Set}^{\Delta^{op}} \rightarrow \text{Mono}(\text{Set}^K)$ is a very-well-behaved reflection.

The same conclusion holds if one replaces Δ by its full subcategory Δ_n , determined by the $n + 1$ objects $[0], [1], \dots, [n]$.

For instance, if $\mathbb{B} = \Delta_0^{op} \cong \mathbf{1}$, with $K : \Delta_0^{op} \rightarrow \Delta^{op}$ the full inclusion, one gets the non-trivial (i.e. $\mathcal{E}'_I$ strictly contained in $\mathcal{E}_I$) monotone-light factorization for simplicial sets via ordered simplicial complexes (cf. [9]).

5. General Results for Sub-reflections from Models

In the present paper, a *sketch* consists of a category $\mathbb{A}$, together with a chosen set of distinguished cones of the form

$$\lambda : \Delta(a) \rightarrow L,$$

where $\Delta : \mathbb{A} \rightarrow \mathbb{A}^{\mathbb{J}}$ is the diagonal functor, $\mathbb{J}$ is a small category and $L : \mathbb{J} \rightarrow \mathbb{A}$ is a functor ($\mathbb{J}, L, a, \lambda$ vary from one distinguished cone to another).

$\check{\mathbb{A}}(\mathbb{S})$ shall denote the full subcategory of the functor category $\hat{\mathbb{A}}(\mathbb{S}) = \mathbb{S}^{\mathbb{A}}$ determined by the models of such a sketch, that is, whose objects are the functors $M : \mathbb{A} \rightarrow \mathbb{S}$ for which $M\lambda = M\Delta(a) \rightarrow ML$ is a limiting cone in $\mathbb{S}$, for each distinguished cone in the sketch.

If $\mathbb{S} = \text{Set}$ then we will simply write $\hat{\mathbb{A}} (= \hat{\mathbb{A}}(\text{Set}) = \text{Set}^{\mathbb{A}})$ and $\check{\mathbb{A}} (= \check{\mathbb{A}}(\text{Set}))$.

If there is a factorization system $(\mathcal{E}, \mathcal{M})$ on $\mathbb{S}$, then $(\hat{\mathcal{E}}, \hat{\mathcal{M}}) = (\mathcal{E}^{\mathbb{A}}, \mathcal{M}^{\mathbb{A}})$ will denote the factorization system which is the uplifting of $(\mathcal{E}, \mathcal{M})$ to $\hat{\mathbb{A}}(\mathbb{S})$ (cf. Lemma 2.16(b)).

Lemma 5.1. *Let $(\mathcal{E}, \mathcal{M})$ be a factorization system on a category $\mathbb{S}$.*

Consider any natural transformation $\varphi : F \rightarrow G : \mathbb{J} \rightarrow \mathbb{S}$, where both functors F, G are supposed to have limits in $\mathbb{S}$, with cones $(\text{Lim} F, (\lambda_i^F)_{i \in \mathbb{J}})$ and $(\text{Lim} G, (\lambda_i^G)_{i \in \mathbb{J}})$; and s is the object obtained by the $(\mathcal{E}, \mathcal{M})$ -factorization of the canonical morphism $\text{Lim} \varphi = \mu \circ \eta : \text{Lim} F \rightarrow s \rightarrow \text{Lim} G$.

Consider also the functor $D : \mathbb{J} \rightarrow \mathbb{S}$ obtained by factorizing $\varphi = n \cdot f : F \rightarrow D \rightarrow G$ using the factorization system $(\hat{\mathcal{E}}, \hat{\mathcal{M}})$, which is the uplifting of the factorization system $(\mathcal{E}, \mathcal{M})$ to $\hat{\mathbb{S}}(\mathbb{J}) = \mathbb{S}^{\mathbb{J}}$ (i.e., using the $(\mathcal{E}, \mathcal{M})$ -factorization for each component $\varphi_i = n_i \circ f_i : F(i) \rightarrow D(i) \rightarrow G(i)$, $i \in \mathbb{J}$).

If $\text{Lim} D$ exists in $\mathbb{S}$, then it is isomorphic to the object s , provided $\text{Lim} f : \text{Lim} F \rightarrow \text{Lim} D$ is in $\mathcal{E}$, with $n \cdot f : F \rightarrow D \rightarrow G$ the $(\hat{\mathcal{E}}, \hat{\mathcal{M}})$ -factorization of φ .

Proof. It is known that $\text{Lim } n : \text{Lim } D \rightarrow \text{Lim } G$ is in $\mathcal{M}$, since every $n_i \in \mathcal{M}$, $i \in \mathbb{J}$ (cf. [1, Proposition 2.2(d)]). Hence, $\text{Lim } D \cong s$, by definition of factorization system (cf. [1, §2.8]). $\square$

The following Corollary 5.2 is intended for future applications, detailing sufficient conditions which are not for the most needed in the next applications to n-categories (in the present paper only the item (a) is relevant).

Corollary 5.2. *Under the conditions of the previous Lemma 5.1:*

- (a) *if $\mathbb{J} = i \xrightarrow{u} k \xleftarrow{v} j$, being $\text{Lim } F$ and $\text{Lim } G$ pullbacks, and if the pullback $\text{Lim } D$ exists, then it is isomorphic to the object s obtained by the $(\mathcal{E}, \mathcal{M})$ -factorization of the canonical morphism $\varphi_i \times \varphi_j = \mu \circ \eta : \text{Lim } F \rightarrow s \rightarrow \text{Lim } G$, provided $f_i, f_j \in \mathcal{E}'$ (the largest subclass of $\mathcal{E}$ stable under pullbacks) and $n_k, \varphi_k \in \text{Mono}(\mathbb{S})(\Leftarrow \varphi_k \in \mathcal{M} \cap \text{Mono}(\mathbb{S}))$, with $\text{Mono}(\mathbb{S})$ the class of monomorphisms of $\mathbb{S}$;*
- (b) *if $\mathbb{J}$ is the discrete category generated by the finite set $\{1, 2, \dots, n\}$, ($n \geq 0$), being $\text{Lim } F = \prod_{i=1}^n F(i)$ and $\text{Lim } G = \prod_{i=1}^n G(i)$ n -products, then the n -product $\text{Lim } D = \prod_{i=1}^n D(i)$ is isomorphic to the object s obtained by the $(\mathcal{E}, \mathcal{M})$ -factorization of the canonical morphism $\prod_{i=1}^n \varphi_i = \mu \circ \eta : \prod_{i=1}^n F(i) \rightarrow \prod_{i=1}^n G(i)$, provided that $\mathbb{S}$ has pullbacks and the class $\mathcal{E}$ of morphisms of $\mathbb{S}$ is stable under pullbacks ($\mathcal{E} = \mathcal{E}'$);*
- (c) *if $\mathbb{J} = \Downarrow (= 0 \xrightarrow{u} \xrightarrow{v} 1)$, being $\text{Lim } F$ and $\text{Lim } G$ equalizers, and if the equalizer $\text{Lim } D$ exists, then it is isomorphic to the object s obtained by the $(\mathcal{E}, \mathcal{M})$ -factorization of the canonical morphism $\mu \circ \eta : \text{Lim } F \rightarrow s \rightarrow \text{Lim } G$, provided the binary products $F(0) \times F(1)$, $D(0) \times D(1)$, $G(0) \times G(1)$ exist, $n_0 \times n_1 : D(0) \times D(1) \rightarrow G(0) \times G(1)$, $\varphi_0 \times \varphi_1 : F(0) \times F(1) \rightarrow G(0) \times G(1)$ are monomorphisms in $\mathbb{S}$ ($\Leftarrow n_0, n_1, \varphi_0, \varphi_1 \in \text{Monos}(\mathbb{S})$), and $f_0 \in \mathcal{E}'$;*
- (d) *if $\mathbb{S}$ has finite limits, then $\text{Lim } D$ is isomorphic to the object s obtained by the $(\mathcal{E}, \mathcal{M})$ -factorization of the canonical morphism $\text{Lim } \varphi = \mu \circ \eta : \text{Lim } F \rightarrow s \rightarrow \text{Lim } G$, provided $\mathbb{J}$ is finite, $\varphi_i : F(i) \rightarrow G(i)$ and $n_i : D(i) \rightarrow G(i)$ are monomorphisms for every $i \in \mathbb{J}$, and $\mathcal{E} = \mathcal{E}'$.*

Proof. (a) Consider the commutative diagram

$$\begin{array}{ccccc}
 \text{Lim}F & \xrightarrow{\lambda_j^F} & & & F(j) \\
 \downarrow \lambda_i^F & \searrow f_i \times f_j & & \nearrow \lambda_j^D & \downarrow f_j \\
 & & \text{Lim}D & \xrightarrow{\lambda_j^D} & D(j) \\
 & & \downarrow \lambda_i^D & \searrow n_i \times n_j & \downarrow n_j \\
 & & & & \text{Lim}G \xrightarrow{\lambda_j^G} G(j) \\
 & & & \downarrow \lambda_i^G & \downarrow Gv \\
 F(i) & \xrightarrow{f_i} & D(i) & \xrightarrow{n_i} & G(i) \xrightarrow{Gu} G(k).
 \end{array}$$

The outside square and the middle square are pullback diagrams, since φ_k and n_k are monomorphisms by hypothesis: $Gu \circ n_i \circ f_i = Gu \circ \varphi_i = \varphi_k \circ Fu$ and $Gv \circ n_j \circ f_j = Gv \circ \varphi_j = \varphi_k \circ Fv$; $Gu \circ n_i = n_k \circ Du$ and $Gv \circ n_j = n_k \circ Dv$.

According to the properties of a factorization system (cf. [1, §2.8] and last Lemma 5.1), one needs to show that $n_i \times n_j \in \mathcal{M}$ and $f_i \times f_j \in \mathcal{E}$ in order that $s \cong \text{Lim}D$. That is obviously so, since $n_i \times n_j \in \mathcal{M}$ by the general property of prefactorization systems stated in [1, Proposition 2.2(d)], and $f_i, f_j \in \mathcal{E}'$.

(b) If $n = 0$ and hence $F = \emptyset = G$, then the $(\mathcal{E}, \mathcal{M})$ -factorization of the canonical morphism is $t = t = t$, where $t = \text{Lim}\emptyset = \text{Lim}F = \text{Lim}G$ is a terminal object of $\mathbb{S}$.

If $n = 1$ it is also obvious, since $\text{Lim}F = F(1)$, $\text{Lim}G = G(1)$ and $\text{Lim}D = D(1)$ ($\varphi_1 = n_1 \circ f_1 : F(1) \rightarrow D(1) \rightarrow G(1)$).

If $n = 2$, then $\text{Lim}F = F(1) \times F(2)$, $\text{Lim}G = G(1) \times G(2)$ are binary products. Consider the commutative diagram

$$\begin{array}{ccccc}
D(1) & \xleftarrow{\lambda_1^D} & D(1) \times D(2) & \xrightarrow{\lambda_2^D} & D(2) \\
\downarrow n_1 & \swarrow p_2^1 & \downarrow q_1 & \searrow q_2 & \downarrow n_2 \\
& & P(1) & & P(2) \\
& & \downarrow p_1^1 & & \downarrow p_1^2 \\
G(1) & \xleftarrow{\lambda_1^G} & G(1) \times G(2) & \xrightarrow{\lambda_2^G} & G(2),
\end{array}$$

where the three squares $\lambda_1^G \circ p_1^1 = n_1 \circ p_2^1$, $\lambda_2^G \circ p_1^2 = n_2 \circ p_2^2$ and $p_1^1 \circ q_1 = p_1^2 \circ q_2$ are pullback diagrams, $p_2^i \circ q_i = \lambda_i^D$ and $n_1 \times n_2 = p_1^i \circ q_i$ ($i = 1, 2$). It was just shown that $\text{Lim} D = D(1) \times D(2)$ exists and can be obtained using pullbacks, which are supposed to exist in the category $\mathbb{S}$.

Consider now the next commutative diagram

$$\begin{array}{ccccc}
F(1) & \xleftarrow{\lambda_1^F} & F(1) \times F(2) & \xrightarrow{\lambda_2^F} & F(2) \\
f_1 \downarrow & & f_1 \times f_2 \downarrow & & \downarrow f_2 \\
D(1) & \xleftarrow{\lambda_1^D} & D(1) \times D(2) & \xrightarrow{\lambda_2^D} & D(2) \\
n_1 \downarrow & & n_1 \times n_2 \downarrow & & \downarrow n_2 \\
G(1) & \xleftarrow{\lambda_1^G} & G(1) \times G(2) & \xrightarrow{\lambda_2^G} & G(2),
\end{array}$$

where $n_1 \times n_2 \in \mathcal{M}$ (cf. Proposition 2.2(d) in [1]). Then, one has just to show that $f_1 \times f_2 \in \mathcal{E}$, which is so since the binary product $F(1) \times F(2)$ may be obtained via pullbacks (from the product diagram of $D(1) \times D(2)$, exactly in the same way as we showed that $D(1) \times D(2)$ exists using the product diagram of $G(1) \times G(2)$), and the class of morphisms $\mathcal{E}$ of $\mathbb{S}$ is stable under pullbacks ($\mathcal{E} = \mathcal{E}'$).

For $n \geq 3$, the statement follows immediately by iteration of binary products.

(c) The proof of this item is just an adaptation of that of item (b), using the well known fact that any equalizer diagram is a pullback diagram if there is a binary product of the two objects involved other than the limit object.

Consider the commutative diagram

$$\begin{array}{ccccc}
\text{Lim} F & \xrightarrow{\lambda_0^F} & F(0) & & \\
\downarrow \lambda_0^F & \searrow \text{Lim} f & & \downarrow f_0 & \\
& & \text{Lim} D & \xrightarrow{\lambda_0^D} & D(0) \\
& & \downarrow \lambda_0^D & \searrow \text{Lim} n & \downarrow n_0 \\
& & & & \text{Lim} G \xrightarrow{\lambda_0^G} G(0) \\
& & & & \downarrow \lambda_0^G \\
F(0) & \xrightarrow{f_0} & D(0) & \xrightarrow{n_0} & G(0) \xrightarrow{(1, Gu)} G(0) \times G(1) \\
& & & & \downarrow (1, Gv)
\end{array}$$

The outside square and the middle square are pullback diagrams, since $\varphi_0 \times \varphi_1$ and $n_0 \times n_1$ are monomorphisms by hypothesis: $(1, Gu) \circ n_0 \circ f_0 = (1, Gu) \circ \varphi_0 = \varphi_0 \times \varphi_1 \circ (1, Fu)$ and $(1, Gv) \circ n_0 \circ f_0 = (1, Gv) \circ \varphi_0 = \varphi_0 \times \varphi_1 \circ (1, Fv)$; $(1, Gu) \circ n_0 = n_0 \times n_1 \circ (1, Du)$ and $(1, Gv) \circ n_0 = n_0 \times n_1 \circ (1, Dv)$.

According to the properties of a factorization system (cf. the references in the proof of (a)), one needs to show that $\text{Lim} f = f_0 \times_{G(0) \times G(1)} f_0 \in \mathcal{E}$ in order that $s \cong \text{Lim} D$. That is so, since $f_0 \in \mathcal{E}'$.

(d) The proof follows from the proof of item (c), since any limit of a functor $X : \mathbb{J} \rightarrow \mathbb{S}$ can be presented canonically using products and an equalizer of two specific morphisms φ^X, γ^X in $\mathbb{S}$ (cf. [8, §V.2]). Just consider the following commutative diagram,

$$\begin{array}{ccccc}
\text{Lim}F & \xrightarrow{\lambda_{\prod_{i \in \mathbb{J}} F(i)}^F} & \prod_{i \in \mathbb{J}} F(i) & & \\
\downarrow \lambda_{\prod_{i \in \mathbb{J}} F(i)}^F & \searrow \text{Lim}f & \downarrow \prod_{i \in \mathbb{J}} f_i & & \\
& & \text{Lim}D & \xrightarrow{\lambda_{\prod_{i \in \mathbb{J}} D(i)}^D} & \prod_{i \in \mathbb{J}} D(i) \\
& & \downarrow \lambda_{\prod_{i \in \mathbb{J}} D(i)}^D & \searrow \text{Lim}n & \downarrow \prod_{i \in \mathbb{J}} n_i \\
& & & & \text{Lim}G & \xrightarrow{\lambda_{\prod_{i \in \mathbb{J}} G(i)}^G} & \prod_{i \in \mathbb{J}} G(i) \\
& & & & \downarrow \lambda_{\prod_{i \in \mathbb{J}} G(i)}^G & & \downarrow (1, \gamma^G) \\
\prod_{i \in \mathbb{J}} F(i) & \xrightarrow{\prod_{i \in \mathbb{J}} f_i} & \prod_{i \in \mathbb{J}} D(i) & \xrightarrow{\prod_{i \in \mathbb{J}} n_i} & \prod_{i \in \mathbb{J}} G(i) & \xrightarrow{(1, \varphi^G)} & \prod_{i \in \mathbb{J}} G(i) \times \prod_u G(\text{cod } u),
\end{array}$$

which is the adaptation of the diagram in the proof of item (c), to the three equalizers arising from the given limits. Remark that, according to item (c), it is needed that $\prod_{i \in \mathbb{J}} \varphi_i \times \prod_u \varphi_{\text{cod } u} \in \text{Mono}(\mathbb{S})$ ($\Leftarrow \varphi_i \in \text{Mono}(\mathbb{S})$, $i \in \mathbb{J}$), $\prod_{i \in \mathbb{J}} n_i \times \prod_u n_{\text{cod } u} \in \text{Mono}(\mathbb{S})$ ($\Leftarrow n_i \in \text{Mono}(\mathbb{S})$, $i \in \mathbb{J}$), and $\prod_{i \in \mathbb{J}} f_i \in \text{Mono}(\mathbb{S}) \in \mathcal{E}$ ($\Leftarrow \mathcal{E} = \mathcal{E}'$ and the set of objects of $\mathbb{J}$ is finite; cf. item (b) above). $\square$

Proposition 5.3. Consider an adjunction $\text{Ran}_K \vdash \text{Set}^K : \hat{\mathbb{A}} = \text{Set}^{\mathbb{A}} \rightarrow \hat{\mathbb{B}} = \text{Set}^{\mathbb{B}}$, as in Theorem 4.5.

Let $\check{\mathbb{A}}$ be the full subcategory determined by the models of a (fixed) sketch (cf. the beginning of this section 5), such that $\text{Ran}_K(M \circ K) \in \check{\mathbb{A}}$ and the canonical map $\text{Lim}(M \circ L) \rightarrow \text{Lim}(D \circ L)$ is always a surjection, where D is the presheaf obtained by the (Surjections, Injections)-factorization of the unit morphism $\theta_M : M \rightarrow \text{Ran}_K(M \circ K)$, for every $M \in \check{\mathbb{A}}$ and for every distinguished $L : \mathbb{J} \rightarrow \mathbb{A}$.

Then, there is a reflection with stable units $\check{H} \vdash \check{I} : \check{\mathbb{A}} \rightarrow \check{\mathbb{M}}$ which is a subreflection of $H \vdash I : \hat{\mathbb{A}} \rightarrow \mathbb{M} = \text{Mono}(\text{Set}^K)$, and $(\mathcal{E}_{\check{I}}, \mathcal{M}_{\check{I}}) = (\mathcal{E}_I \cap \text{Mor}(\check{\mathbb{A}}), \mathcal{M}_I \cap \text{Mor}(\check{\mathbb{A}}))$. Here, $\check{\mathbb{M}}$ is the full subcategory of $\mathbb{M}$ determined by the models of the given sketch on $\hat{\mathbb{A}}$, $(\mathcal{E}_{\check{I}}, \mathcal{M}_{\check{I}})$ and $(\mathcal{E}_I, \mathcal{M}_I)$ are the reflective factorization systems associated respectively to the reflections $\check{H} \vdash \check{I}$ and $H \vdash I$, and $\text{Mor}(\check{\mathbb{A}})$ is the class of all morphisms in $\check{\mathbb{A}}$.

Proof. As, by hypothesis, for every $M \in \check{\mathbb{A}}$,

$$\text{Ran}_K(MK) \in \check{\mathbb{A}}$$

($\Leftrightarrow \text{Ran}_K(MK) \in \check{\mathbb{M}}$ by Remark 2.5), then $I(M) \in \check{\mathbb{M}}$ according to Lemma 5.1 just above: consider, in the statement of Lemma 5.1, $F = M \circ L$, $G = \text{Ran}_K(MK) \circ L : \mathbb{J} \rightarrow \mathbb{A} \rightarrow \text{Set} = \mathbb{S}$, $\varphi = \theta_M \circ L$. Therefore, $\check{\mathbb{M}}$ is indeed a full reflective subcategory of $\check{\mathbb{A}}$.

Let $(\mathcal{E}_{\check{I}}, \mathcal{M}_{\check{I}})$ be the reflective factorization system associated to $\check{I} \dashv \check{H}$. It is known that $\gamma : M \rightarrow N$ is in $\mathcal{E}_{\check{I}}$ if and only if $\check{I}\gamma = I\gamma$ is an isomorphism (cf. [1, §3.1]), therefore $\mathcal{E}_{\check{I}} = \mathcal{E}_I \cap \text{Mor}(\check{\mathbb{A}})$. As $\check{\mathbb{A}}$ is closed under limits in $\hat{\mathbb{A}}$ (using the property of interchange of limits, cf. [8, §IX.2]), it follows that $\mathcal{E}'_I \cap \text{Mor}(\check{\mathbb{A}}) \subseteq \mathcal{E}'_{\check{I}}$. Hence, $\check{I} \dashv \check{H}$ has stable units because $I \dashv H$ has stable units (the units $\check{\eta}_M = \eta_M \in \mathcal{E}'_I$, $M \in \check{\mathbb{A}}$). Being $\check{H} \vdash \check{I}$ a reflection with stable units it is also simple, and then $\mathcal{M}_{\check{I}} = \mathcal{M}_I \cap \text{Mor}(\check{\mathbb{A}})$, because $\check{\mathbb{A}}$ is closed under limits in $\hat{\mathbb{A}}$ and by the characterization of the morphisms in $\check{\mathbb{M}}$ in a simple reflection (cf. Theorem 4.1 in [2] or [1, §3.5]). $\square$

Examples 5.4. Consider $\mathbb{P} = \Delta_3^{op}$ the dual of the truncated simplicial category generated by the diagram (where $0 \leq i \leq k+1$ for d_i^k and $0 \leq j \leq k$ for s_j^k , $k \in \{0, 1, 2\}$)

$$[3] \begin{array}{c} \xrightarrow{d_i^2} \\ \xleftarrow{s_j^2} \end{array} [2] \begin{array}{c} \xrightarrow{d_i^1} \\ \xleftarrow{s_j^1} \end{array} [1] \begin{array}{c} \xrightarrow{d_i^0} \\ \xleftarrow{s_0^0} \end{array} [0]$$

and the usual relations (cf. Examples 4.6 and [8, §VII.5]; notice that this example could as well be worked out using the dual of the entire simplicial category $\mathbb{A} = \Delta^{op}$)¹.

Let $K : \Delta_0^{op} \rightarrow \Delta_3^{op} = \mathbb{P}$ be the inclusion functor ($\Delta_0^{op} \cong 1$). Since $\{[0]\}$ is a cogenerating class for Δ_3^{op} (because $[0]$ is a generator in Δ), then, by Theorem 4.5, there is a reflection $H \vdash I : \hat{\mathbb{P}} = \text{Set}^{\mathbb{P}} \rightarrow \mathbb{M} = \text{Monos}(\text{Set}^K)$ which is very-well-behaved (i.e., with stable units and a monotone-light factorization). $\hat{\mathbb{P}}$ could be notated Smp_3 , the category of truncated simplicial sets.

This reflection induces a subreflection with stable units $\check{H} \vdash \check{I} : \check{\mathbb{P}} \rightarrow \check{\mathbb{M}}$, for the sketch whose models $M : \mathbb{P} \rightarrow \text{Set}$ are those functors such that $Md_1^0 \circ Md_0^1 = Md_0^0 \circ Md_2^1$ and $Md_2^1 \circ Md_0^2 = Md_0^1 \circ Md_3^2$ are pull-back squares, according to Proposition 5.3. Indeed, $\text{Ran}_K \circ \text{Set}^K(S) \in \check{\mathbb{P}}$

¹It follows from the Lemma in [8, §VII.5] that: truncating the dual of the simplicial category gives the same result as using the diagram above and the usual relations to generate Δ_3^{op} .

for every $S \in \hat{\mathbb{P}}$ (not only for every $S \in \check{\mathbb{P}}$): it is easy to check that $Ran_K(SK)([n]) = Lim((([n] \downarrow K) \xrightarrow{Q} \Delta_0^{op} \xrightarrow{SK} Set) = S_0^{n+1}$, the $n+1$ -power of $S_0 = S([0])$, $0 \leq n \leq 3$; hence, $Ran_K(SK)$ is an indiscrete category (a connected equivalence relation). $\check{\mathbb{P}}$ can be identified with Cat , the category of all small categories and $\check{\mathbb{M}}$ with the category $Preord$ of all preordered sets.

Remark that Lemma 5.1 is needed so that $Md_2^1 \circ Md_0^2 = Md_0^1 \circ Md_3^2$ is a pullback, but $Md_1^0 \circ Md_0^1 = Md_0^0 \circ Md_2^1$ being a pullback is an immediate consequence of Corollary 5.2(a) (indeed, there $\varphi_k = \theta_{M,[0]}$ is the identity).

The reflection $Cat \rightarrow Preord$ was showed to be very-well-behaved in [13], where the monotone-light factorization was presented and showed to be non-trivial ($\mathcal{E}'_I \neq \mathcal{E}_I$).

The existence of a monotone-light factorization is a consequence of Theorem 2.9: for every category $M \in \check{\mathbb{P}}$, there is an effective descent morphism $p : E \rightarrow M$ in $\hat{\mathbb{P}}$ which is also an effective descent morphism in $\check{\mathbb{P}} = Cat$ (cf. the characterization of effective descent morphisms in Cat given in [5]); p is the canonical presentation of the presheaf M , where E is the coproduct of representable functors which are linear orders, and so E is obviously also a linear order (a special case of a preorder).

5.1 Discussion of the very-well-behaved reflection from 2-categories into 2-preorders

This and the following last subsection are presented in a lighter way than the rest of the paper. We expect to be more precise in a future paper.

The example discussed in this subsection was fully studied in [14], with a slightly different presentation from the one given here. In order to relate to that paper [14], so that the interested reader may have an easier task consulting it, we shall call *precategories* the objects of $\hat{\mathbb{P}} = Set^{\mathbb{P}}$ in Examples 5.4 just above, and the diagram there which generates the category $\mathbb{P}$ is now called a *precategory diagram*, with $P_i = [i]$ ($0 \leq i \leq 3$).

Consider the category $2\mathbb{P}$ generated by the following 2-*precategory* diagram,

$$\begin{array}{ccccccc}
& \xrightarrow{\quad} & & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\
P_{43} & \xleftarrow{\quad} & P_{42} & \xleftarrow{\quad} & P_{41} & \xleftarrow{\quad} & P_0 \\
\downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow 1_{P_0} \\
P_{33} & \xleftarrow{\quad} & P_{32} & \xleftarrow{\quad} & P_{31} & \xleftarrow{\quad} & P_0 \\
\downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow 1_{P_0} \\
P_{23} & \xleftarrow{\quad} & P_{22} & \xleftarrow{\quad} & P_{21} & \xleftarrow{\quad} & P_0 \\
\downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow 1_{P_0} \\
P_3 & \xleftarrow{s_j^2} & P_2 & \xleftarrow{s_j^1} & P_1 & \xleftarrow{s_0^0} & P_0, \\
& \xrightarrow{d_i^2} & & \xrightarrow{d_i^1} & & \xrightarrow{d_i^0} &
\end{array}$$

in which:

- each one of the four horizontal diagrams

$$\begin{array}{ccccc}
& \xrightarrow{d_i^{l2}} & & \xrightarrow{d_i^{l1}} & & \xrightarrow{d_i^{l0}} \\
P_{l3} & \xleftarrow{s_j^{l2}} & P_{l2} & \xleftarrow{s_j^{l1}} & P_{l1} & \xleftarrow{s_0^{l0}} & P_{l0}
\end{array}$$

is a precategory diagram (remark that, in the 2-precategory diagram, for $l = 1$, the notation is simplified in order to be similar to the one in Examples 5.4), and each one of the four vertical diagrams

$$\begin{array}{ccccccc}
& \xrightarrow{\check{d}_i^{l2}} & & \xrightarrow{\check{d}_i^{l1}} & & \xrightarrow{\check{d}_i^{l0}} \\
P_{4(l-1)} & \xleftarrow{\check{s}_j^{l2}} & P_{3(l-1)} & \xleftarrow{\check{s}_j^{l1}} & P_{2(l-1)} & \xleftarrow{\check{s}_0^{l0}} & P_{1(l-1)}
\end{array}$$

is also a precategory diagram (the rightmost one is the trivial one with all morphisms identities, as portrayed in the 2-precategory diagram), $l \in \{1, 2, 3, 4\}$;

- in the obvious sense given in [14], the $\check{s}_j^{lk}$ provide precategory morphisms vertically, and the s_j^{lk} provide precategory morphisms horizontally, $l \in \{1, 2, 3, 4\}$; similarly for the $\check{d}_i^{lk}$ and d_j^{lk} .

Let $K : \mathbb{P} \rightarrow 2\mathbb{P}$ be the inclusion functor of $\mathbb{P}$ in $2\mathbb{P}$

(recall, from Examples 5.4, that $\mathbb{P}$ is generated by

$$P_3 \xrightleftharpoons[s_j^2]{d_i^2} P_2 \xrightleftharpoons[s_j^1]{d_i^1} P_1 \xrightleftharpoons[s_0^0]{d_i^0} P_0 \text{ and the usual relations}).$$

$\{P_3 = P_{13}, P_2 = P_{12}, P_1 = P_{11}, P_0 = P_{10}\}$ is a cogenerating class for $2\mathbb{P}$, because it obviously is a generating class for $2\mathbb{P}^{op}$. Being K a full inclusion, it is well known that $Ran_K \vdash Set^K$ is a full reflection (cf. Corollary 3 in [8, §X.3]), hence Ran_K can be seen as the inclusion of the category $\hat{\mathbb{P}}$ in $2\hat{\mathbb{P}}$. Then, by Theorem 4.5, there is a very-well-behaved reflection

$$H \vdash I : 2\hat{\mathbb{P}} = Set^{2\hat{\mathbb{P}}} \rightarrow 2\mathbb{M} = Monos(Set^K).$$

This reflection induces a subreflection having stable units

$$\check{H} \vdash \check{I} : 2\check{\mathbb{P}} \rightarrow 2\check{\mathbb{M}},$$

for the sketch whose models are those functors M such that $Md_1^{l0} \circ Md_0^{l1} = Md_0^{l0} \circ Md_2^{l1}$, $Md_2^{l1} \circ Md_0^{l2} = Md_0^{l1} \circ Md_3^{l2}$ and $M\check{d}_1^{l0} \circ M\check{d}_0^{l1} = M\check{d}_0^{l0} \circ M\check{d}_2^{l1}$, $M\check{d}_2^{l1} \circ M\check{d}_0^{l2} = M\check{d}_0^{l1} \circ M\check{d}_3^{l2}$ are pullback squares for every horizontal and vertical precategory diagram respectively, $l \in \{1, 2, 3, 4\}$. Indeed, it is clear then that $Ran_K \circ Set^K(M) \in 2\hat{\mathbb{P}}$ for every $M \in 2\hat{\mathbb{P}}$, so that the conditions of Proposition 5.3 hold.

$2\check{\mathbb{P}}$ can be identified with $2Cat$, the category of all 2-categories, and $2\check{\mathbb{M}}$ with the category $2Preord$ of all 2-preorders (cf. the beginning of §5 in [14]).

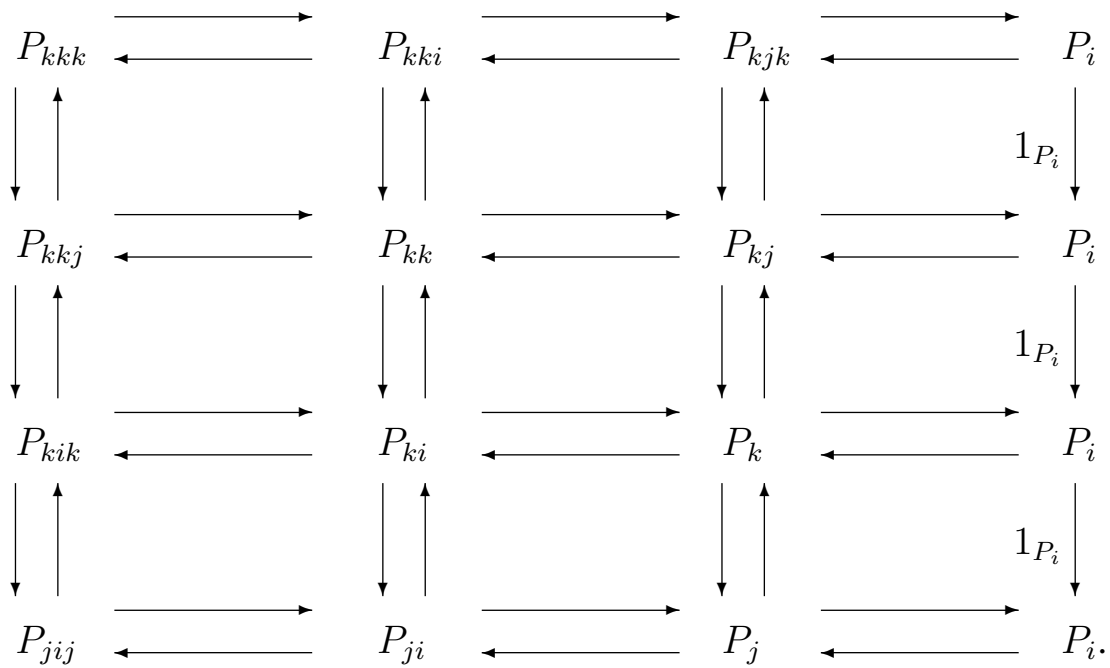
The reflection $2Cat \rightarrow 2Preord$ was showed to be very-well-behaved in [14], where the monotone-light factorization was displayed and showed to be non-trivial ($\mathcal{E}'_{\check{I}} \neq \mathcal{E}_{\check{I}}$). The existence of a monotone-light factorization is a consequence of Theorem 2.9: for every 2-category $M \in 2\check{\mathbb{P}}$, there is an effective descent morphism $p : E \rightarrow M$ in $2\hat{\mathbb{P}}$ which is also an effective descent morphism in $2\check{\mathbb{P}} = 2Cat$.

In [14], the proof that $\check{H} \vdash \check{I} : 2Cat \rightarrow 2Preord$ has stable units is much more complicated than in here. In that paper, one used to the full extent the results in [11] to prove directly that the reflection $\check{H} \vdash \check{I}$ has stable units, while in the present paper this reflection inherits this property (although, to do so, a small theory had to be developed).

5.2 Discussion of the very-well-behaved reflection from n -categories into n -preorders

This subsection generalizes the example of the previous subsection, whose results are subsumed here. Anyway, the somewhat repeating of subjects helps the exposition and discussion of these examples, which can be displayed using different theoretical paths.

Consider the category $n\mathbb{P}$ generated by all the 2-precategory diagrams below (cf. [15, §10] and last subsection 5.1), for all integers i, j and k such that $0 \leq i < j < k \leq n$,



The number $\frac{(n-1)n(n+1)}{6}$ of these 2-precategory diagrams can easily be obtained by using the counting principle. Remark that these $\frac{(n-1)n(n+1)}{6}$ diagrams are not independent since they share common objects and arrows (every arrow between two common objects), and that if $n = 2$ there is only one 2-precategory diagram.

Let $K : (n-1)\mathbb{P} \rightarrow n\mathbb{P}$ be the inclusion functor of $(n-1)\mathbb{P}$ in $n\mathbb{P}$.

Being K a full inclusion, it is well known that $\text{Ran}_K \vdash \text{Set}^K$ is a full

reflection (cf. Corollary 3 in [8, §X.3]), hence Ran_K can be seen as the inclusion of the category $(n-1)\mathbb{P}$ in $\hat{n}\mathbb{P}$.

The objects of $(n-1)\mathbb{P}$ constitute a cogenerating class for $n\mathbb{P}$, because it obviously is a generating object for $(n\mathbb{P})^{op}$. Then, by Theorem 4.5, there is a very-well-behaved reflection

$$H \vdash I : \hat{n}\mathbb{P} = Set^{\hat{n}\mathbb{P}} \rightarrow n\mathbb{M} = Monos(Set^K).$$

This reflection induces a subreflection having stable units

$$\check{H} \vdash \check{I} : \check{n}\mathbb{P} \rightarrow \check{n}\mathbb{M},$$

for the sketch whose models are those functors M such that $Md_1 \circ Md_0 = Md_0 \circ Md_2$ and $Md_2 \circ Md_0 = Md_0 \circ Md_3$ are pullback squares for every horizontal and vertical precategory diagram (to understand this rather scanty notation, please cf. last subsection 5.1). Indeed, it is clear then that $Ran_K \circ Set^K(M) \in \check{n}\mathbb{P}$ for every $M \in \check{n}\mathbb{P}$, so that the conditions of Proposition 5.3 hold.

$\check{n}\mathbb{P}$ can be identified with $nCat$ (cf. [15, §10]), the category of all n -categories, and $\check{n}\mathbb{M}$ with the category $nPreord$ of all n -preorders (cf. the beginning of §12 in [15]).

The reflection $nCat \rightarrow nPreord$ was showed to be very-well-behaved in [15], where the monotone-light factorization was displayed and showed to be non-trivial ($\mathcal{E}'_I \neq \mathcal{E}_I$), for every $n \geq 2$. The existence of a monotone-light factorization is a consequence of Theorem 2.9: for every n -category $M \in \check{n}\mathbb{P}$, there is an effective descent morphism $p : E \rightarrow M$ in $\hat{n}\mathbb{P}$ which is also an effective descent morphism in $\check{n}\mathbb{P} = nCat$.

In [15], the proof that $\check{H} \vdash \check{I} : nCat \rightarrow nPreord$ has stable units was made using a quite different path from the one used here. In that paper, the category of n -categories was considered in the context of $\mathcal{V}$ -categories; it is known that, considering $\mathcal{V} = Set$ and then iterating, n -categories are obtained (see [7]). There, beginning with the well-behaved reflection $Cat \rightarrow Preord$ and using the *change of enriching category* (see [3]), it was possible to show that the derived reflection from n -categories into n -preorders had stable units, while in the present paper this reflection inherits this property from larger reflections in which it is contained.

It is also relevant to state that the path in the context of $\mathcal{V}$ –*categories*, in order to establish the stable units property, could be replaced by the process given in [11] which was used for 2-categories in [14].

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